Data Linkages and Incentives*

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Abstract

Many firms, such as banks and insurers, condition their level of service on a consumer's perceived "quality," for instance their creditworthiness. Increasingly, firms have access to consumer segmentations derived from auxiliary data on behavior, and can link outcomes across individuals in a segment for prediction. How does this practice affect consumer incentives to exert (socially-valuable) effort, e.g. to repay loans? We show that the impact of a linkage on behavior depends crucially on whether the linkage reflects quality (via correlations in types) or a shared circumstance (via common shocks to observed outcomes).

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1 Introduction

A bank receives a credit card application from a consumer, Alice. In addition to traditional sources of information about Alice—e.g. her repayment history—the bank also learns from a data broker that Alice has been classified as an active gym member, based on usage statistics from gyms and geolocation data from her devices. Since other individuals in the "active gym member" segment have consistently paid off their credit card balances, the bank provides Alice with a high credit limit.

This fictional story is increasingly descriptive of actual industry practice. In particular, data brokers now regularly aggregate personal data about consumers, and use this data to identify segments of consumers with similar characteristics and likely behaviors. The range of consumer segments is diverse (see Appendix A for a list of examples): Some focus on broad lifestyle patterns, e.g. "Bible Lifestyle," "Soccer Mom," "Exercise–Sporty Living," "New Age/Organic Lifestyle." Others group consumers based on narrower preferences or activities, e.g. "Outdoor/Hunting & Shooting," "Leans Left," or "Fitness Enthusiast." Still others reflect recent life events, such as getting married, buying a home, or sending a child to college. These segmentations are passed onto companies such as banks and insurance agencies, who have begun using them to decide what level of service to provide a consumer.[1]

Identifying similarities across consumers can help an organization to better predict their behaviors. But categorization also reshapes incentives for effort, e.g. paying off credit card balances promptly and driving more attentively. If Alice knows that the bank evaluates her not only based on her own repayment history, but also on the repayment histories of other individuals in her category, how does that change her incentives to exercise financial prudence? Since effort in such contexts can be socially valuable, it is important to understand the externalities created when organizations use data from one individual to inform predictions about others.

[1]For example: In 2008, the subprime lender CompuCredit was revealed to have reduced credit lines based on visits to various "red flag" establishments, including marriage counselors and nightclubs (see `https://www.bloomberg.com/news/articles/2008-06-18/your-lifestyle-may-hurt-your-credit`); some health insurance companies acquire predictions from data brokers like LexisNexis for anticipated health costs (see `https://www.pbs.org/newshour/health/why-health-insurers-track-when-you-buy-plus-size-clothes-or-binge-watch-tv`); the car insurance company Allstate recently filed a patent for adjusting insurance rates based on routes and historical accident patterns (see `https://www.usatoday.com/story/money/personalfinance/2016/11/14/route-risk-patent--car-insurance-rate-price/93287372`). And perhaps most strikingly, China's "social credit" system determines whether an individual is a good citizen based on detailed attributes ranging from the size of their social network to how often they play video games (see `https://foreignpolicy.com/2018/04/03/life-inside-chinas-social-credit-laboratory`).

To study this question, we fix an exogenously given consumer segment (e.g. all individuals identified as "active gym members"), and consider the incentives of those agents. Our framework is a multiple-agent version of the classic career concerns model (Holmström 1982a). Each agent has an unknown *type* (e.g. creditworthiness), which a principal (a bank) would like to predict. Agents choose whether to opt-in to interaction with the principal (sign up for a credit card). The principal observes an outcome (the agent's past repayment behavior) from each agent who opts in, which is informative about the agent's underlying type, but also perhaps about the types of others in his segment. The agent can manipulate his own outcome via costly effort (exercising financial prudence). We say that a *data linkage* exists when a principal bases its prediction of the agent's type on the outcomes of other participating agents, in addition to the agent's own data. Data linkages create an informational externality across agents.

We study how these data linkages affect both consumers' willingness to opt in to relationships with a principal, as well as their incentives to exert effort in those relationships. The answer depends crucially on the way in which each agent's outcome is informative about others in his segment. We contrast two distinct models of linkages between agents. One model of data linkage relates to agents' *quality*. Within such segments, the types that the principal cares about forecasting are correlated. In the credit example, this relationship may be a lifestyle pattern (e.g. "Frequent Flier," "Fitness Enthusiast," "Exercise—Sporty Living") or personal characteristic (e.g. "Working-class Mom," "Spanish Speaker"). Lending outcomes from consumers in such a segment can be used to better predict repayment for other similar individuals. The second model of linkage relates to a common *circumstance*. In such segments, agents experience common shocks to their outcomes. For example, drivers who commute on the same roads to work are exposed to similar variations in local road conditions, e.g. construction or bad weather. Auto insurers can use their aggregated outcomes to estimate the sizes of these shocks and de-bias observed accident rates.

We show that these two models of data linkages result in starkly different equilibrium behaviors. When a linkage across individuals in a segment relates to quality, linking the agents ensures full participation, but depresses the amount of effort they exert (relative to a benchmark in which the principal does not observe the linkage). In contrast, when the identified linkage is about a shared circumstance, linking agents reduces participation rates, but raises the amount of effort that participating agents exert. These results are robust to agent uncertainty about the details of the segment—such as the strength of the correlation between outcomes and the size of the population—requiring only that agents know whether their linkage to other agents relates to quality or circumstance.

The main intuition is as follows. In the quality linkage model, consumer data are *sub-*

stitutes—for instance, observation of repayment rates for other active gym members helps a bank to learn an average long-run repayment rate for this segment, reducing the marginal informativeness of any given borrower's outcome. Thus, when the principal aggregates data from consumers within the segment, distortion of one's outcome via extra effort has a smaller influence on the principal's perception about one's type. In contrast, in the circumstance linkage model, consumer data are *complements*—for instance, observation of accident rates for other drivers who take the same roads to work is informative about the size of the "road condition shock" on insurance claims. Each driver's de-biased claims rate is more informative about his type, so the value to exerting effort to improve the observed outcome becomes larger under the linkage.

These comparative statics have direct implications for consumer payoffs from participation: In our model, as in Holmström (1982a), the principal correctly infers the equilibrium level of effort and can de-bias observed outcomes.[2] Since effort is costly, higher equilibrium effort necessarily means lower payoffs for agents. (If effort is socially valuable, this need not imply lower *social* welfare, as we discuss below.) Thus, in the quality linkage model, agent participation decisions are strategic complements: Participation by one agent improves the payoffs to participation for other agents by decreasing equilibrium effort. We show existence of a unique equilibrium in which all agents choose to opt-in to interaction with the principal. In contrast, in the circumstance linkage model, participation creates a negative externality on other agents by increasing equilibrium effort. For small populations, there is again an all opt-in equilibrium, while for large populations, agents must mix over entry in the unique symmetric equilibrium.

Our results can be readily mapped into real-world consequences for specific applications. If a credit card issuer links borrowers in a segment, where agents in that segment have correlated qualities—e.g. because of a common lifestyle or level of financial literacy—agents will retain their credit cards, but this linkage will cause agents in that segment to exercise less financial prudence, increasing default rates. A firm that values responsible financial behavior may therefore prefer to commit *not* to use big data analytics for identifying such correlations, instead forecasting each agent's behavior in isolation.

On the other hand, if agents within the segment share a common circumstance—e.g. having recently moved or started a new job—then the linkage induces participating agents to exert higher effort. Depending on the size of the segment, these linkages may also cause

[2] Frankel and Kartik (2019b) introduces uncertainty in the ability of agents to manipulate outcomes, so that the principal cannot perfectly de-bias the impact of effort. In such settings a reduction in incentives for effort improves the precision of forecasts, creating a tradeoff for the principal when effort and precise forecasts are both valuable.

some agents to withdraw from interaction with the principal, e.g. by canceling their credit cards. For sufficiently small segments, the organization enjoys full participation and higher effort, in which case the data linkage unambiguously benefits the principal. On the other hand, if the segment is sufficiently large, then the linkage results in higher effort but lower rates of participation. Whether the organization benefits from use of this linkage then depends on how it trades off between these two goals.

Our results also have important implications for the impact of data linkages on social welfare. We first show that in both models and for all segment sizes, equilibrium actions are inefficiently low relative to the first-best (extending a result established in Holmström (1982a) for Gaussian signals). We then compare equilibrium outcomes against a "no linkages" benchmark in which the principal is permitted to use only an agent's *own* past data to predict his type. When agents are connected by a quality linkage, aggregation of data across agents invariably leads to a reduction in social welfare. In contrast, when agents share common circumstances, the welfare implications of data linkages depend on the number of agents within the segment, and can go either way. These results suggest that the type of data being used to link agents is a crucial determinant of the welfare effect of data linkages.

Finally, we use our model and results to comment on a current policy debate regarding whether firms should have proprietary ownership of their data, or if this data should be shared across an industry (as for example recently recommended by the European Commission).[3] To study this issue, we extend our model to a setting with many principals (firms) who compete over consumers via a monetary reward for participation. Under *proprietary data*, firms observe only the outcomes of agents who participate with them, while under *data sharing*, the outcomes of all agents are shared across firms. We show that regardless of whether agents are linked by quality or circumstance, data sharing leads to an increase in consumer welfare. Market forces play a key role in this result: in particular, if firms were not able to optimize their participation rewards, then the welfare implications of data sharing *would* depend on the nature of the linkage.

[3]As reported in European Commission (2020): "[T]he Commission will explore the need for legislative action on issues that affect relations between actors in the data-agile economy to provide incentives for horizontal data sharing across sectors." Such action might "support business-to-business data sharing, in particular addressing issues related to usage rights for co-generated data...typically laid down in private contracts. The Commission will also seek to identify and address any undue existing hurdles hindering data sharing and to clarify rules for the responsible use of data (such as legal liability). The general principle shall be to facilitate voluntary data sharing."

1.1 Related Literature

Our paper contributes to an emerging literature regarding the welfare consequences of data markets and algorithmic scoring. This literature has tackled several important social questions, such as whether predictive algorithms discriminate (Chouldechova 2017; Kleinberg et al. 2017); how to protect consumers from loss of privacy (Acquisti et al. 2015; Dwork and Roth 2014; Eilat et al. 2019; Fainmesser et al. 2019); how to price data (Agarwal et al. 2019; Bergemann, Bonatti, and Smolin 2018); whether seller or advertiser access to big data harms consumers (Gomes and Pavan 2018; Jullien et al. 2018); and how to aggregate big data into market segments or consumer scores (Bonatti and Cisternas 2019; Elliott and Galeotti 2019; Hidir and Vellodi 2019; Ichihashi 2019; Yang 2019). There is additionally a growing literature about strategic interactions with machine learning algorithms: see Eliaz and Spiegler (2018) on the incentives to truthfully report characteristics to a machine learning algorithm, and Olea et al. (2018) on how economic markets select certain models for making predictions over others.

In particular, Acemoglu et al. (2019) and Bergemann, Bonatti, and Gan (2019) also consider externalities created by social data. Different from us, these papers study data sharing in environments where consumers may sell their data. In Bergemann, Bonatti, and Gan (2019), one agent's information improves a firm's ability to price-discriminate against other agents, which can decrease consumer surplus. In Acemoglu et al. (2019), agents value privacy, and thus information collected about one agent imposes a direct negative externality on other agents when types are correlated. The externality of interest in the present paper is how information provided by other agents reshapes incentives to exert costly *effort*. As we show, this externality can be positive or negative—in particular, when agents are connected by a quality linkage, their equilibrium payoffs turn out to be *increasing* in other agents' participation.

At a theoretical level, our paper builds on the career concerns model of Holmström (1982a), the classic framework for analyzing the role of reputation-building in motivating effort. The interaction of this incentive effect with informational externalities from other agents' behavior is the main focus of our analysis. The literature following Holmström (1982a) has largely focused on signal extraction about a single agent's type in dynamic settings,[4] while we are interested in the externalities of social data in a multiple-agent setting. Our paper is most closely related to Dewatripont et al. (1999), which studies how auxiliary

[4] A small set of papers, e.g. Auriol et al. (2002), study career concerns in a multiple-agent setting. These papers typically look at effort externalities instead of informational externalities. One exception is Meyer and Vickers (1997), which considers the impact of adding an additional agent with correlated outcomes in the context of a ratchet effect model with incentive contracts.

data impacts agents' incentives for effort. That paper considers the externality of a single exogenous auxiliary signal, while we endogenize the auxiliary data as information from other players, who strategically decide whether or not to provide data. Thus, the number of auxiliary signals is determined in equilibrium, and may also be uncertain; this requires comparison of equilibrium actions across various information structures.

Our circumstance linkage model, in which the principal uses outcomes from some agents to help de-bias the outcomes of other agents, is reminiscent of the relational contracting and tournaments literatures (Green and Stokey 1983; Holmström 1982b; Lazear and Rosen 1981; Meyer and Vickers 1997; Shleifer 1985). In these papers, the observable output of each agent depends both on the agent's effort as well as on a common shock experienced by all agents. In such environments the relative output of an agent is a more precise signal of effort than the absolute output. Thus the principal may be able to extract more effort through rewarding good relative outcomes rather than good absolute outcomes. Although we do not consider a contracting environment here, similar forces in our model permit the principal to extract more effort from agents when their outcomes are related by correlated shocks.

Finally, our paper contributes to work on strategic manipulation of information. Recent papers in this category include: Frankel and Kartik (2019a) and Ball (2019), which characterize the degree to which a principal with commitment power should link his decision to a manipulated signal about the agent's type; Hu et al. (2019), which shows that heterogeneous manipulation costs across different social groups can lead to inequities in outcomes; and Georgiadis and Powell (2019), which studies optimal information acquisition for a designer setting a wage contract. Our paper contributes to this literature by exploring the role of correlations across data in an individual's incentives to manipulate an observed outcome.

2 Model

A single principal interacts with $N < \infty$ agents, who have been identified as belonging to the same consumer segment. Each agent i has a type $\theta_i \in \mathbb{R}$, which is unknown to all parties (including agent i) and is commonly believed to be drawn from the distribution F_θ with mean $\mu > 0$ and finite variance $\sigma_\theta^2 > 0$.[5] Types are drawn symmetrically but may not be independent across agents.

As in the classic career concerns model of Holmström (1982a), each agent's payoffs are

[5] None of our results would change if we gave the principal access to additional privately observed covariates for use in forecasting. Specifically, we could allow θ_i to be decomposable as $\theta_i = \theta_i^0 + \Delta\theta_i$, where θ_i^0 is commonly unobserved with mean 0 while $\Delta\theta_i$ is an idiosyncratic type shifter, independent of θ_i^0 with mean μ, which is privately observed by the principal.

increasing in the principal's perception of his type, and the agent can exert costly effort to influence an outcome realization that the principal observes (Section 2.2). Different from Holmström (1982a), we introduce a preliminary stage at which the agent first chooses whether to opt-in or out of interaction with the principal (Section 2.1), and—most importantly—we allow the principal to aggregate the outcomes of multiple agents for prediction (Section 2.3).

The model unfolds over three periods, with opt-in/out decisions made in period $t = 0$, effort exerted in period $t = 1$, and forecasts of each agent's type based on outcomes updated in period $t = 2$.

2.1 Period 0—Opt-In/Opt-Out

At period $t = 0$, each agent i first chooses whether to *opt-in* or *opt-out* of an interaction with the principal, where this decision is observed by the principal, but not by other agents. Opting out yields a payoff that we normalize to zero. The set of agents who opt-in is denoted $\mathscr{I}_{\text{opt-in}} \subseteq \{1, \ldots, N\}$.

2.2 Period 1—Choice of Costly Effort to Influence Outcome

In period $t = 1$, each agent $i \in \mathscr{I}_{\text{opt-in}}$ privately chooses a costly effort level $a_i \in \mathbb{R}_+$ to influence an observable outcome. The outcome, S_i, is related to the agent's type and effort level via

$$S_i = \theta_i + a_i + \varepsilon_i,$$

where $\varepsilon_i \sim F_\varepsilon$ is a noise shock with mean $\mathbb{E}[\varepsilon_i] = 0$ and finite variance $\mathbb{E}[\varepsilon_i^2] = \sigma_\varepsilon^2 > 0$. Noise shocks are drawn symmetrically but not necessarily independently across agents. We describe the correlation structure across shocks in Section 2.3. The agent's payoff in this period is

$$R - C(a_i)$$

where $R \in \mathbb{R}$ is a monetary opt-in reward from the principal (possibly negative), and $C(a_i)$ is the cost to choosing effort a_i. We suppose that the cost function is twice continuously differentiable and satisfies $\lim_{a_i \to \infty} C'(a_i) > 1$, $C(0) = C'(0) = 0$, and $C''(a_i) > 0$ for all a_i.

2.3 Period 2—Principal's Forecast of Agent's Type

In a second (and final) period, each agent $i \in \mathscr{I}_{\text{opt-in}}$ receives the principal's forecast of the agent's type θ_i. The principal's forecast is based on the observed outcomes of all agents who have opted-in; thus, agent i's payoff in the second period is

$$\mathbb{E}\left[\theta_i \mid S_j, j \in \mathscr{I}_{\text{opt-in}}\right]. \tag{1}$$

Note that since each agent's effort choice is private, the forecast is based on a conjectured effort choice, which in equilibrium is simply the equilibrium effort level. This payoff is a stand-in for the reputational consequences of the agent's period-1 outcome.[6] Note that the agent's payoff is increasing in the principal's forecast of their type, reflecting the role of θ_i as a quality variable determining average outcomes.

The quantity in (1) depends on the (random) realizations of output; thus, the agent optimizes over his *expectation* of (1). We will discuss this iterated expectation of θ_i in detail in Section 3.1. Finally, the agent's total payoff is the sum of his expected payoffs across the two periods. This timeline is summarized in Figure 1.

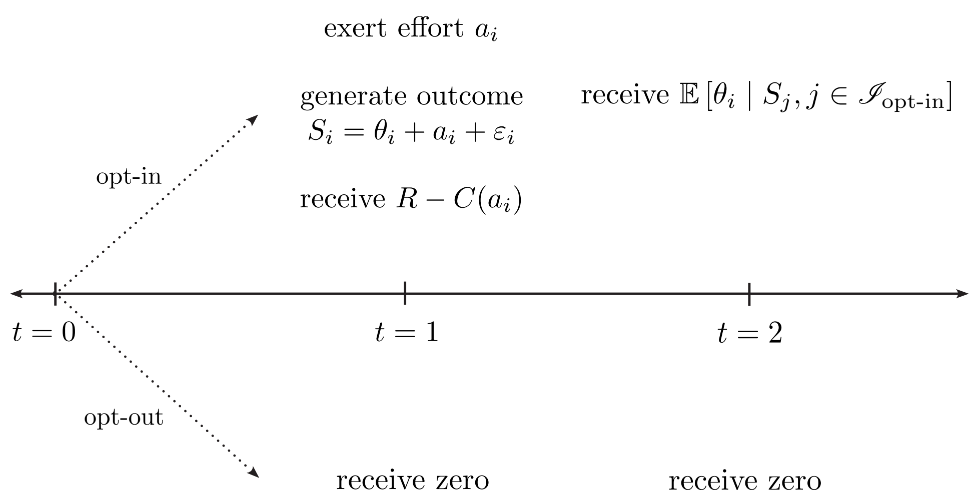

Figure 1: Timeline

So far we have not described how agent outcomes are correlated, a specification which is crucial for computing the posterior expectation in (1). Our main analysis contrasts two kinds of relationships across agents, one in which agents within a segment have related qualities, and another in which they share a related circumstance:

[6] One could view this payoff as representing the agent's payoff in a second-period market where multiple firms compete to serve the agent. Our main results would be unchanged if we allowed the agent's reputational payoff to be any increasing affine transformation of $\mathbb{E}[\theta_i \mid S_j, j \in \mathscr{I}_{\text{opt-in}}]$, accommodating service over multiple future periods and different assumptions regarding market structure and price-setting.

Quality Linkage. Suppose first that agents within the segment have correlated qualities. We model this by decomposing θ_i as

$$\theta_i = \overline{\theta} + \theta_i^{\perp},$$

where $\overline{\theta} \sim F_{\overline{\theta}}$ is a common component of the type and $\theta_i^{\perp} \sim F_{\theta^{\perp}}$ is a personal or idiosyncratic component, with each $\theta_i^{\perp}$ independent of $\overline{\theta}$ and all $\theta_j^{\perp}$ for $j \neq i$. Without loss, we assume $\mathbb{E}[\overline{\theta}] = \mu$ while $\mathbb{E}[\theta_i^{\perp}] = 0$. In contrast, the shocks ε_i are mutually independent.

Circumstance Linkage. Another possibility is that agents within the segment don't have qualities which are intrinsically related, but instead have experienced a shared shock to outcomes. Formally, we suppose that the noise shock can be decomposed as

$$\varepsilon_i = \overline{\varepsilon} + \varepsilon_i^{\perp}$$

where $\overline{\varepsilon} \sim F_{\overline{\varepsilon}}$ is shared across agents and $\varepsilon_i^{\perp} \sim F_{\varepsilon^{\perp}}$ is idiosyncratic, with each $\varepsilon_i^{\perp}$ independent of $\overline{\varepsilon}$ and all $\varepsilon_j^{\perp}$ for $j \neq i$. In contrast, agents' types θ_i are mutually independent.

The distinction between quality and circumstance linkages can be interpreted in at least two ways. One interpretation is that θ_i is the portion of the outcome that is valuable to the principal, while ε_i is a confounder that has an effect on the observed outcome, but is not payoff-relevant. Another interpretation is that the type θ_i is a permanent component of the agent's performance while ε_i is a shock that affects performance only temporarily. Many of our subsequent examples are of the latter form, where ε_i reflects a transient characteristic that affected outcomes in a previous observation cycle, but is no longer present in future interactions. For example, if an agent was pregnant during the determination of S_i, but has since given birth, then the principal should optimally de-noise the "pregnancy effect" from the prior outcome when predicting future behaviors. Throughout the paper, we consider these two models of linkage separately in order to clarify the difference between them.

Note that while the correlation structure across agent outcomes differs in the two models, we will hold the marginal distributions of each agent's type and noise shock fixed across models (see Assumption 2).

2.4 Examples

Commuters and auto-insurers. The principal is an auto-insurer and the agents are commuters. Agent i's type θ_i is a function of his accident risk while driving, with higher-type commuters experiencing a lower risk of accidents while driving to work. Each commuter decides whether to own a car versus commuting via rideshares or public transit. Conditional on owning a car, the commuter then chooses how much effort to exert to drive safely. The insurance company observes his claims rate during an initial enrollment period, and uses that outcome to predict his future claims rates.

Examples of *quality linkage* segments include drivers who share similar commutes to work, e.g. routes primarily through surface streets or via highways, where these routes are discoverable from geolocational data. If we suspect that commutes are stable and that the route taken contributes to the risk of accident, then claims rates for other drivers in the segment are directly informative about the future accident risk for a given driver. Examples of *circumstance linkage* segments include drivers who passed through routes that were previously affected by unusual road or weather conditions. Crucially, these conditions are not expected to persist into the subsequent period.[7] The principal can use claims rates from drivers in this segment to learn the size and direction of the "road shock" or "weather shock," allowing them to de-bias observed accident rates.

Consumers and credit-card issuers. The principal is a bank issuing a credit card and agents are consumers. Agent i's type θ_i is his creditworthiness, with more creditworthy consumers being better able to pay back short-term loans. Each agent decides whether to sign up for a credit card versus making payments by debit card or cash. If an agent signs up for a credit card, he decides how much effort to exert in order to ensure repayment (e.g. by increasing income or avoiding activities that risk financial loss), and the card issuer observes his repayment behavior during an initial enrollment period.

Quality linkages relevant to creditworthiness include lifestyles ("Frequent Flier") and financial sophistication ("Subscriber to Financial Newsletter"), categories which can be revealed by social media usage and online subscription databases. Circumstance linkages include whether a consumer's child was previously attending college (but has since graduated) and whether a family member was previously experiencing a serious illness (but has since improved), as inferred for example from purchasing and travel histories.

2.5 Solution Concept

We study Nash equilibria in which agents choose symmetric participation strategies and pure strategies in effort. Our focus on symmetric participation reflects the ex-ante symmetry of consumers in our model, and their anonymity with respect to one another in most data

[7] If the conditions are persistent, we would consider the consumers instead to be related by a quality linkage.

markets. In the absence of a centralized mechanism, we expect that consumers would find it challenging to coordinate asymmetric participation. Our restriction to equilibria with deterministic effort follows the career concerns literature, and plays an important role in maintaining tractability.[8]

We additionally impose a refinement on out-of-equilibrium beliefs. Since agents choose participation and effort simultaneously in our model, Nash equilibrium puts no restrictions on the principal's inference about effort in the event that an agent unexpectedly enters. We require that if an agent unilaterally deviates to entry, the principal expects that the agent will exert the equilibrium effort choice from a single-agent game with exogenous entry. This refinement mimics sequential rationality in a modified model in which agents make entry and effort decisions sequentially rather than simultaneously. In what follows, we will us the term *equilibrium* without qualification to refer to symmetric equilibria in pure effort strategies satisfying this refinement.

2.6 Distributional Assumptions

We impose several regularity conditions on the distributions $F_{\overline{\theta}}$, $F_{\theta^\perp}$, $F_{\overline{\varepsilon}}$, and $F_{\varepsilon^\perp}$, which are maintained throughout the paper. Assumptions 1 through 4 are purely technical, and ensure that all distributions have full support and are smooth enough for appropriate derivatives of conditional expectations to exist. Assumptions 5 and 6 are substantive, and ensure monotonicity of inferences about latent variables in outcome and sufficiency of the first-order approach for characterizing equilibrium effort.

Assumption 1 (Regularity of densities)**.** *The distribution functions $F_{\overline{\theta}}$, $F_{\theta^\perp}$, $F_{\overline{\varepsilon}}$, $F_{\varepsilon^\perp}$ admit strictly positive, C^1 density functions $f_{\overline{\theta}}$, $f_{\theta^\perp}$, $f_{\overline{\varepsilon}}$, $f_{\varepsilon^\perp}$ with bounded first derivatives on $\mathbb{R}$.*

Assumption 2 (Invariance of marginal densities)**.** *In each model, the distribution functions F_θ and F_ε have density functions f_θ and f_ε satisfying $f_\theta = f_{\overline{\theta}} * f_{\theta^\perp}$ and $f_\varepsilon = f_{\overline{\varepsilon}} * f_{\varepsilon^\perp}$, where $*$ is the convolution operator.*

In each model one half of Assumption 2 is redundant, as in the quality linkage model $\theta_i = \overline{\theta} + \theta_i^\perp$ while in the circumstance linkage model $\varepsilon_i = \overline{\varepsilon} + \varepsilon_i^\perp$. The remaining half of the assumption ensures that θ_i and ε_i have the same marginal distributions across models. The following corollary reflects the fact that convolutions of variables satisfying the properties of Assumption 1 inherit those properties.

[8]When agents mix over effort, then even under the assumptions imposed in Section 2.6 higher output is not guaranteed to lead to higher inferences about types. Depending on the equilibrium distribution of effort, the principal may instead attribute a positive output shock to high realized effort. See Rodina (2017) for further discussion.

Corollary 1. *f_θ and f_ε are strictly positive, C^1, and have bounded first derivatives on $\mathbb{R}$.*

The following assumption ensures that posterior expectations are smooth enough to compute first and second derivatives of an agent's value function, and to compute the marginal impact of a change in one agent's outcome on the forecast of another agent's type. Let $\mathbf{S} = (S_1, ..., S_N)$ be the vector of outcomes for all agents, with $\mathbf{a} = (a_1, ..., a_N)$ the vector of actions for all agents.

Assumption 3 (Regularity of posterior expectations)**.** *For each model, population size N, agent $i \in \{1, ..., N\}$, and outcome-action profile $(\mathbf{S}, \mathbf{a})$:*

- $\frac{\partial}{\partial S_j}\mathbb{E}[\theta_i \mid \mathbf{S}; \mathbf{a}]$ *exists and is continuous in* $\mathbf{S}$ *for every* $j \in \{1, ..., N\}$*,*
- $\frac{\partial^2}{\partial S_i^2}\mathbb{E}[\theta_i \mid \mathbf{S}; \mathbf{a}]$ *exists.*

The following assumption is a slight strengthening of the requirement that the Fisher information of S_i about its common component ($\overline{\theta}$ in the quality linkage model or $\overline{\varepsilon}$ in the circumstance linkage model) be finite. Let $f_{\varepsilon+\theta^\perp} \equiv f_{\theta^\perp} * f_\varepsilon$ and $f_{\theta+\varepsilon^\perp} \equiv f_\theta * f_{\varepsilon^\perp}$.

Assumption 4 (Finite Fisher information)**.** *For each $f \in \{f_{\varepsilon+\theta^\perp}, f_{\theta+\varepsilon^\perp}\}$, there exists a $\overline{\Delta} > 0$ and a dominating function $J : \mathbb{R} \to \mathbb{R}_+$ such that*

$$\left(\frac{1}{\Delta}\frac{f(z-\Delta) - f(z)}{f(z)}\right)^2 \leq J(z)$$

for all $z \in \mathbb{R}$ and $\Delta \in (0, \overline{\Delta})$ and

$$\int J(z) f(z)\, dz < \infty.$$

Roughly, this assumption ensures that finite-difference approximations to the Fisher information are also finite and uniformly bounded as the approximation becomes more precise.[9]

The following assumption imposes enough structure on the distributions of the components of each agent's outcome to ensure that higher outcome realizations imply monotonically higher forecasts of the components of the outcome.

[9] A sufficient condition for Assumption 4 is that $f_{\varepsilon+\theta^\perp}$ and $f_{\theta+\varepsilon^\perp}$ don't vanish at the tails "much faster" than their derivatives: specifically, for each $f \in \{f_{\varepsilon+\theta^\perp}, f_{\theta+\varepsilon^\perp}\}$ there should exist a $K > 0$ and $\overline{\Delta} > 0$ such that:

$$\max_{\varepsilon \in \mathbb{R}, \Delta \in [0, \overline{\Delta}]} \left|\frac{f'(\varepsilon - \Delta)}{f(\varepsilon)}\right| \leq K.$$

This sufficient condition is satisfied, for example, by the t-distribution and the logistic distribution. It is *not* satisfied by the normal distribution, although we show in Appendix O.2.1 using other methods that the normal distribution does satisfy Assumption 4. We are not aware of any commonly-used distributions which violate Assumption 4.

Assumption 5 (Monotone forecasts)**.** *The density functions $f_{\overline{\theta}}$, $f_{\theta^\perp}$, $f_{\overline{\varepsilon}}$, and $f_{\varepsilon^\perp}$ are strictly log-concave.*[10]

One basic property of strictly log-concave functions is that the convolution of two log-concave functions is also strictly log-concave. Thus an immediate corollary of Assumption 5 is the following:

Corollary 2. *f_θ and f_ε are strictly log-concave.*

Assumption 5 implies monotonicity of forecasts for the following reason. In general, given three random variables X, Y, Z such that $X = Y + Z$ and Y and Z are independent, strict log-concavity of the density function of Z is both necessary and sufficient for the distribution of X to satisfy a strict monotone likelihood-ratio property in Y (Saumard and Wellner 2014):

$$\frac{f_{X|Y}(x' \mid y')}{f_{X|Y}(x \mid y')} > \frac{f_{X|Y}(x' \mid y')}{f_{X|Y}(x \mid y)} \quad \text{if and only if} \quad x' > x, y' > y.$$

This monotone likelihood-ratio property is the canonical sufficient condition ensuring monotonicity of the conditional expectation of Y in the observed value of X (Milgrom 1981). Assumption 5 guarantees that the appropriate monotone likelihood-ratio properties are satisfied in our model; see Appendix B.1 for details.

Finally, we assume the cost function is "sufficiently convex" that effort choices satisfying a first-order condition are globally optimal. The assumption is a joint condition on the cost function and the distribution of the outcome, since the required amount of convexity depends on how sensitive the posterior expectation is to the realization of individual outcomes.

Assumption 6 (Sufficient convexity)**.** *There exists a $K \in \mathbb{R}$ such that $C''(x) > K$ for every $x \in \mathbb{R}_+$, and for every population size N and agent $i \in \{1, ..., N\}$, $\frac{\partial^2}{\partial S_i^2}\mathbb{E}[\theta_i \mid \mathbf{S}; \mathbf{a}] \leq K$ for every* $(\mathbf{S}, \mathbf{a})$.

One important set of models satisfying these regularity conditions is Gaussian uncertainty.[11]

Example (Gaussian). For each agent i,

$$\begin{pmatrix} \overline{\theta} \\ \theta_i^\perp \\ \overline{\varepsilon} \\ \varepsilon_i^\perp \end{pmatrix} \sim \mathcal{N}\left(\begin{pmatrix} \mu \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \sigma_{\overline{\theta}}^2 & 0 & 0 & 0 \\ 0 & \sigma_{\theta^\perp}^2 & 0 & 0 \\ 0 & 0 & \sigma_{\overline{\varepsilon}}^2 & 0 \\ 0 & 0 & 0 & \sigma_{\varepsilon^\perp}^2 \end{pmatrix} \right).$$

[10] A function $g > 0$ is *strictly log-concave* if $\log g$ is strictly concave.

[11] The Gaussian versions of our quality and circumstance linkage models represent special cases of the information environment considered in Meyer and Vickers (1997) and Bergemann, Bonatti, and Gan (2019), both of whom allow for correlation between both types and shocks. The Gaussian version of our quality linkage model also corresponds to a symmetric version of the environment considered in Acemoglu et al. (2019).

We verify in Appendix O.2.1 that Assumptions 1 through 5 are all met in this case, and Assumption 6 is satisfied by any strictly concave cost function.

3 Preliminary Results: Exogenous Entry

We begin our analysis by studying a restricted model, in which the number of agents who opt-in is exogenously specified. Without loss, we suppose that all N agents participate.

3.1 Marginal Value of Effort

In equilibrium, agents choose effort such that the marginal impact of effort on the principal's forecast in the second period, which we will refer to as the *marginal value of effort*, equals its marginal cost. Here we define the marginal value of effort and explore its properties.

Fix an equilibrium effort profile $(a_1^*, ..., a_N^*)$. The principal believes that each outcome is distributed $S_i = \theta_i + a_i^* + \varepsilon_i$, and any agent i who chooses the equilibrium effort level a_i^* believes the same. But if some agent i deviates to a non-equilibrium action $a_i \neq a_i^*$, then he knows that his outcome is distributed $S_i = \theta_i + a_i + \varepsilon_i$. This means that the agent's expected period-2 reward (i.e. the agent's expectation of the principal's forecast of his type) is an iterated expectation with respect to two different probability measures over the space of types and outcomes.

Formally, let $\mathbb{E}^\Delta$ denote expectations when agent i chooses effort level $a_i^* + \Delta$. For any profile of realized outcomes $(S_1, \ldots, S_N)$, the principal's expectation of agent i's type is

$$\mathbb{E}^0[\theta_i \mid S_1, \ldots, S_N].$$

If agent i exerts effort $a_i = a_i^* + \Delta$, then his ex-ante expectation of the principal's forecast is

$$\mu_N(\Delta) \equiv \mathbb{E}^\Delta[\mathbb{E}^0[\theta_1 \mid S_1, \ldots, S_N]].$$

Note that if the agent does not distort his effort away from the equilibrium level, then $\mu_N(0) = \mu$, reflecting the usual martingale property of posterior expectations.

When $\Delta \neq 0$, posterior expectations under the principal's beliefs are *not* a martingale from agent 1's perspective: As we show in Appendix C, $\mu_N(\Delta)$ is strictly increasing in Δ. Thus, increasing effort beyond the expected effort level always leads to a higher expected value of the principal's expectation.[12] The agent's incentives to distort effort away from its

[12]Kartik et al. (2019) showed that if two agents with differing priors update beliefs in response to signals about an unknown state, the more optimistic agent expects the other's expectation of the state to increase. Our Lemma C.1 complements this result, finding an analogous effect when two agents share a common prior but disagree about the correlation between the state and the signal.

equilibrium level are characterized by the *marginal value of effort* $MV(N)$, which is defined as

$$MV(N) \equiv \mu'_N(0).$$

Our notation reflects the fact that $\mu'_N(0)$, thus also $MV(N)$, is independent of the equilibrium effort levels $a_1^*, ..., a_N^*$, due to the additive dependence of outcomes on effort.

Example. In the Gaussian model described in Section 2.6, an agent who exerts effort $a = a^* + \Delta$ expects the principal's forecast of his type to be

$$\mu_N(\Delta) = \mu + \beta(N) \cdot \Delta$$

for a function $\beta(N)$ that is independent of Δ and a. See Online Appendix O.2.2 for the closed-form expression for $\beta(N)$ (which differs depending on whether we assume a quality linkage or circumstance linkage). The existence of closed-form expressions, as well as linearity of $\mu_N(\Delta)$, are particular to Gaussian uncertainty, although independence with respect to the equilibrium effort level is general. The marginal value of effort $MV(N) = \mu'_N(0)$ is then simply the constant slope $\beta(N)$ in this Gaussian setting.

Throughout, we use $MV_Q(N)$ and $MV_C(N)$ to denote the marginal value functions in the quality linkage and circumstance linkage models, dropping the subscript when a statement holds in both models.

3.2 Equilibrium Effort

Since agents are symmetric, they share the same marginal value and marginal cost of effort. There is therefore a unique effort level $a^*(N)$ satisfying each agent's equilibrium first-order condition

$$MV(N) = C'(a^*(N)) \tag{2}$$

equating the marginal value of effort $MV(N)$ with its equilibrium marginal cost $C'(a^*(N))$. This condition is both necessary and sufficient to ensure that—when the principal expects all agents to exert effort $a^*(N)$—each agent's optimal effort choice is indeed $a^*(N)$. The unique equilibrium of the exogenous-entry model then entails choice of

$$a^*(N) = C'^{-1}(MV(N)) \tag{3}$$

by every agent. When we wish to denote equilibrium effort in the quality linkage or the circumstance linkage model specifically, we will write $a_Q^*(N)$ or $a_C^*(N)$ respectively. Note that $a_Q^*(1) = a_C^*(1)$; that is, the equilibrium action is the same in the single-agent version of both models.

3.3 Key Lemma: Population Size and the Marginal Value of Effort

We now characterize how the number of participating agents impacts each agent's incentives to exert effort. This comparative static plays a key role in characterizing equilibrium in the full model.

Lemma 1. *The marginal value of effort exhibits the following comparative static in population size:*

(a) $MV_Q(N)$ *is strictly decreasing in* N *and* $\lim_{N\to\infty} MV_Q(N) > 0$.

(b) $MV_C(N)$ *is strictly increasing in* N *and* $\lim_{N\to\infty} MV_C(N) < 1$.

That is, the marginal value of effort declines in the number of agents in the quality linkage model, and increases in the circumstance linkage model. Since C' is strictly increasing, it is immediate from this lemma and (3) that the equilibrium actions $a^*(N)$ display the same comparative statics.[13]

Proposition 1. *Equilibrium effort in the exogenous entry model exhibits the following comparative static in population size:*

(a) $a^*_Q(N)$ *is strictly decreasing in* N *and* $\lim_{N\to\infty} a^*_Q(N) > 0$.

(b) $a^*_C(N)$ *is strictly increasing in* N *and* $\lim_{N\to\infty} a^*_C(N) < \infty$.

The key to this result is understanding how the number of observations N impacts the sensitivity of the principal's forecast of θ_i to the realization of S_i. All else equal, the stronger the dependence of this forecast on i's outcome, the stronger the incentive to manipulate its distribution. In the circumstance linkage model, other agents' data (which are informative about the common component of the noise term $\overline{\varepsilon}$) complements agent i's outcome, improving its marginal informativeness. Thus, the larger N is, the more weight the principal puts on i's outcome in its forecast of θ_i. This force incentivizes effort. In the limit as $N \to \infty$, the principal learns $\overline{\varepsilon}$ perfectly and can de-bias the outcomes accordingly, so the incentives for agent i to exert effort are the same as in a single-agent model with $S_i = \theta_i + \varepsilon_i^{\perp}$.

By contrast, in the quality linkage model other agents' data (which are informative about the common part of the type $\overline{\theta}$) substitutes for i's signal; thus, the larger N is, the less weight the principal puts on the realization of i's outcome in its forecast of θ_i. This force de-incentivizes effort. In the limit as $N \to \infty$, the principal can extract $\overline{\theta}$ perfectly from

[13] Meyer and Vickers (1997) establish the same comparative static in a Gaussian setting with up to two agents; see their Proposition 1.

the outcomes of other agents but retains uncertainty about $\theta_i^\perp$, so manipulation of S_i is still valuable. Specifically, the marginal value of effort is the same as in a single-agent model with $S_i = \theta_i^\perp + \varepsilon_i$.

Although this intuition is straightforward, we do not in general have access to the distribution of the principal's posterior expectation in closed form, so we cannot directly quantify the "strength" of the posterior expectation's dependence on the outcome S_i. Moreover, although it is straightforward to show that the sequence of functions $\mu_N(\Delta)$ converge *pointwise* to a limiting function $\mu_\infty(\Delta)$, the rates of this convergence may vary across Δ. Since we are interested in the limiting marginal value $\lim_{N\to\infty} MV(N) = \lim_{N\to\infty} \mu'_N(0)$, we need the stronger property of *uniform* convergence of $\mu_N(\Delta)$ around $\Delta = 0$. In Appendix C.2.2, we show that the expected impact of increasing effort by Δ, i.e. $\mu_N(\Delta) - \mu_N(0)$, can be bounded by an expression that shrinks (for Part (a)) or grows (for Part (b)) in N uniformly in Δ.[14] This establishes that the marginal value of deviating from equilibrium effort at finite N, $\mu'_N(0)$, indeed converges to the marginal value of effort in the limiting model, $\mu'_\infty(0)$, which we can separately characterize.

4 Main Results

We now return to the main model, where the agents who participate (and thus the segment size N from the previous section) are endogenously determined.

4.1 Equilibrium

In equilibrium, the principal correctly de-biases the impact of effort on observed outcomes. The agent's expected payoff in the second period is thus the prior mean μ, no matter the equilibrium effort level. Therefore opt-in is (weakly) optimal as part of an equilibrium strategy if and only if the agent's equilibrium action a^* satisfies

$$R + \mu - C(a^*) \geq 0.$$

[14] An implication of Lemma 1 is that as $N \to \infty$, the agent's expectation of the principal's forecast converges to the agent's own expectation of his type; that is, μ. This implication has the flavor of the classic Blackwell and Dubins (1962) result on merging of opinions, which says that if two agents have different prior beliefs which are absolutely continuous with respect to one another, then given sufficient information, their posterior beliefs must converge. The difference is that the Blackwell and Dubins (1962) result demonstrates almost-sure convergence, while we are interested in l_1-convergence under a shifted measure—that is, whether the agent's expectation of the principal's expectation converges to the agent's own expectation given sufficient data, where the agent and principal use different priors. Neither of these two notions of convergence directly imply the other.

We impose the following assumption, which guarantees that agents would find it optimal to opt-in when no other agents are present in the segment. This restricts attention to settings in which a functioning market existed prior to identification of linkages across consumers.

Assumption 7 (Individual Entry). *$R + \mu \geq C(a^*(1))$, where $a^*(1)$ is the equilibrium effort in the exogenous-entry game with a single agent (as defined in (2) with $N = 1$).*

In light of Assumption 7, there exists no equilibrium (respecting the refinement introduced in Section 2.5) featuring no entry. This is because in any no-entry equilibrium, an agent deviating to entry and choosing effort $a^*(1)$ would receive a payoff of $R + \mu - C(a^*(1)) > 0$ given that the principal expects the agent to exert effort $a^*(1)$ following such a deviation.

Our main results characterize how the equilibrium implications of quality and circumstance linkages differ:

Theorem 1. *In the quality linkage model, there is a unique equilibrium for all population sizes N. In this equilibrium, each agent opts-in and chooses effort $a^*_Q(N)$.*

Theorem 2. *In the circumstance linkage model, there is a unique equilibrium for all population sizes N. There exists an $N^* \in \{1, 2, ...\} \cup \{\infty\}$ such that:*

- *If $N \leq N^*$, each agent opts-in and chooses effort $a^*_C(N)$,*
- *If $N > N^*$, each agent opts-in with probability $p(N) \in (0, 1)$ and chooses effort $a^{**} \in [a^*_C(N^*), a^*_C(N^* + 1))$. The effort level a^{**} is independent of N, while the opt-in probability $p(N)$ is strictly decreasing in N and satisfies $\lim_{N\to\infty} p(N) = 0$.*

The threshold N^* is increasing in R, and is finite for all R sufficiently small.

The equilibrium actions characterized in Theorems 1 and 2 are depicted in Figure 2.

When the segment size is small, Assumption 7 ensures that opting-in is strictly profitable for all agents in each model, and so the equilibrium effort levels $a^*_Q(N)$ and $a^*_C(N)$ are the same as in the previous section. Thus, the equilibrium effort levels inherit the properties described in Proposition 1. As the population size grows, opting-in becomes increasingly attractive in the quality linkage model, since equilibrium effort $a^*_Q(N)$ decreases in N. As a result, all agents participate no matter how large the population. But in the circumstance linkage model, effort $a^*_C(N)$ increases in N and so participation becomes less attractive as the population of entering agents grows. If N is large enough that the total cost of participation $C[a^*(N)]$ exceeds the expected reward $R+\mu$, then full participation cannot be an equilibrium.

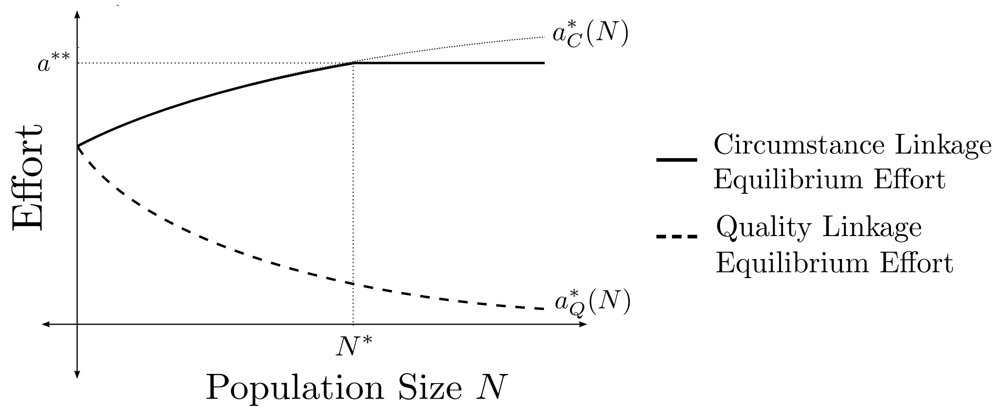

Figure 2: The relationship between population size and equilibrium effort

We let N^* denote the largest N for which $R + \mu \geq C[a^*(N)]$. Then for any $N > N^*$, agents randomize over entry in equilibrium.[15]

In this mixed equilibrium, agents must enter at a rate $p(N) < 1$ and exert an effort level a^{**} so as to satisfy two conditions:

1. Agents are indifferent over entry:

$$R + \mu = C(a^{**}),$$

2. The marginal value of distortion equals its marginal cost:

$$\mathbb{E}\left[MV(1+\widetilde{N}) \;\middle|\; \widetilde{N} \sim \text{Bin}(N-1, p(N))\right] = C'(a^{**})$$

The entry condition pins down the action level a^{**}, which is independent of the population size. The entry rate $p(N)$ is then pinned down by the requirement that the expected marginal value of effort must equal the marginal cost when agents who enter take action level a^{**}. Since the expected marginal value of effort rises with the number of entering agents, $p(N)$ must drop with N to equilibrate marginal values and costs.[16]

[15]If the opt-in reward R is large enough, it may be that $N^* = \infty$ and all agents enter no matter how large the population, as even the limiting effort level for very large populations is worth incurring for the large entry reward. The value N^* is finite whenever R is not too large.

[16]In general, this probability $p(N)$ is not the same as the probability $p^*(N)$ satisfying $MV\left(1 + p^*(N) \cdot (N-1)\right) = C'(a^{**})$, i.e. the opt-in probability such that equilibrium effort is a^{**} given *deterministic* entry of $p^*(N) \cdot (N-1)$ other agents. In the Gaussian setting (and we suspect more generally) $MV(N)$ is a concave function of N, implying that uncertainty in the number of entrants increases the equilibrium rate of entry.

4.2 Welfare Implications

We now analyze the welfare implications of the equilibrium outcomes derived in Section 4.1. Following Holmström (1982a), we consider outcomes to represent socially valuable surplus generated by service provision, while effort is socially costly. In addition, we consider the forecast $\mathbb{E}[\theta_i \mid S_j, j \in \mathscr{I}_{\text{opt-in}}]$ to reflect surplus that the agent receives, e.g. through future service. These factors contribute to social surplus *only* for participating agents, since surplus is not generated by agents who opt-out. Meanwhile, we take the reward R to represent a monetary transfer, which affects the split of surplus but not the amount generated.[17]

For any symmetric strategy profile (p, a) chosen by a population of N agents, where p is the opt-in probability and a is an action choice, we define total expected welfare to be

$$\begin{aligned} W(p, a, N) &= \mathbb{E}\left[\sum_{i=1}^{N} \mathbb{1}(\text{opt-in}) \times [S_i + \mathbb{E}(\theta_i \mid S_j, j \in \mathscr{I}_{\text{opt-in}}) - C(a)]\right] \\ &= pN \cdot (a + 2\mu - C(a)). \end{aligned} \tag{4}$$

Total welfare is divided between the principal and agents as follows: the principal receives the outcome S_i and pays a reward R to every participating agent i, yielding expected profits

$$\Pi(p, a, N) = pN \cdot (a + \mu - R).$$

Meanwhile every participating agent receives reward R and the reputational payoff $\mathbb{E}[\theta_i \mid S_j, j \in \mathscr{I}_{\text{opt-in}}]$, and incurs effort cost $C(a)$. Total consumer welfare is therefore

$$CS(p, a, N) = pN \cdot (R + \mu - C(a)).$$

Note that $W(p, a, N) = \Pi(p, a, N) + CS(p, a, N)$, so all surplus goes to either the principal or one of the agents.

We consider how each of these welfare measures compares to a "no data linkages" benchmark in which the principal does not observe the linkage across agents, and uses only agent i's outcome S_i to predict their type θ_i. That is, the principal's forecast is $\mathbb{E}(\theta_i \mid S_i)$. In equilibrium in this benchmark, each agent opts-in (by Assumption 7), and chooses effort level

$$a_{NDL} \equiv a^*(1) \tag{5}$$

i.e. the action that would be taken for a population of size 1. (Recall that this action is the same for both linkage models.) In a similar spirit to Assumption 7, we assume that serving agents is profitable absent a linkage:

[17] In Section 6.3 we consider how results change if improved prediction also contributes to social welfare.

Assumption 8 (Profitable market)**.** $a^*(1) + \mu > R$.

This assumption ensures that a functioning market existed prior to linkages becoming available, and that the principal would not prefer to drop out rather than serve the market.

4.2.1 Consumer welfare

Consumer welfare depends only on the action each agent is induced to take upon entry, and not on equilibrium entry rates. This is because agents randomize over entry only when opting-in and -out yield the same payoff. So consumer welfare can be computed as if every agent entered and exerted the equilibrium effort level, and this welfare is declining in effort. Therefore consumer welfare drops under any quality linkage and rises under any circumstance linkage, no matter the population size.

4.2.2 Principal profits

Principal profits are rising in effort, and also in the participation rate whenever per-agent profits are positive. When agents within a segment have correlated quality, Theorem 1 indicates that use of the linkage for prediction (increasing the effective population size from 1 to N) will lead to depressed effort by agents without affecting participation, thus reducing firm profits relative to the no-linkage benchmark. Firms may therefore prefer to commit *not* to use big data analytics for forecasting outcomes based on such linkages.

On the other hand, when agents experience shared circumstances (that affect current-period outcomes but are not reflective of underlying quality), Theorem 2 shows that use of the linkage will boost agent effort but may reduce participation. For small segments, firms benefit from the effort boost, and the linkage is profitable. However, for sufficiently large segments the effect of dampened participation outweighs this benefit (since $p(N) \to 0$ as $N \to \infty$ but effort levels are bounded), and the linkage becomes unprofitable.

4.2.3 Social surplus

While firm profits are always increasing in effort and consumer welfare is always decreasing, social welfare is non-monotone in effort. Each participating agent generates a surplus of

$$a + 2\mu - C(a),$$

which is maximized at the unique effort level a_{FB} satisfying $C'(a_{FB}) = 1$. Since $\mu > 0$, surplus is strictly positive at this effort level, and so aggregate surplus is maximized when all agents enter and exert effort a_{FB}.

We first show that equilibrium actions are below the first-best action in both models no matter how many agents participate. This result implies that, fixing the level of participation, linkages which boost effort improve social welfare.

Lemma 2. *For every population size N, equilibrium effort is inefficiently low in both models:*

$$a^*(N) < a_{FB}.$$

As N increases:

- *Effort in the circumstance linkage model $a^*_C(N)$ becomes more efficient but is bounded below the efficient level:* $\lim_{N\to\infty} a^*_C(N) < a_{FB}$.
- *Effort in the quality linkage model $a^*_Q(N)$ becomes less efficient.*

Recall that the equilibrium action a^* satisfies $C'(a^*) = MV(N)$ while the first-best action a_{FB} satisfies $C'(a_{FB}) = 1$. The lemma is proved by demonstrating that $MV(N) < 1$ in both models for all N. Intuitively, some effort is always dissipated, since the realization of the outcome is noisy, so the principal's forecast of θ_i moves less than 1-to-1 with the outcome. This result generalizes a classic result from Holmström (1982a), which demonstrated that $a^*(1) < a_{FB}$ in the case of Gaussian random variables.

The following proposition builds on the previous result and compares $W_{NDL}(N)$, $W_Q(N)$, and $W_C(N)$, which respectively denote social welfare under the no-linkage benchmark, a quality linkage, and a circumstance linkage.

Proposition 2. *For every $N > 1$,*

$$W_Q(N) < W_{NDL}(N).$$

There exists a population threshold $\overline{N}$ such that

$$W_{NDL}(N) < W_C(N)$$

for all $1 < N < \overline{N}$ while

$$W_C(N) < W_{NDL}(N)$$

for all $N > \overline{N}$.

For all populations with $N \geq 2$ agents, quality linkages lead to a reduction in social welfare. This follows directly from Lemma 2: Since there is full entry in the no-data linkages benchmark as well as in the quality linkage equilibrium, the welfare comparison is completely determined by the relative sizes of the equilibrium actions, which are ranked $a_{NDL}(N) = a^*_Q(1) > a^*_Q(N)$.

In contrast, under a circumstance linkage, the comparison depends on the population size N. In small populations, all agents opt-in, so again the action comparison completely determines welfare. Since $a_{NDL}(N) = a^*_C(1) < a^*_C(N)$, data linkages leads to an improvement in social welfare. In large populations, depressed entry dominates and results in lower social welfare despite increased effort levels from participating agents. (Both regimes exist whenever the population threshold N^* above which agents randomize over entry is finite and larger than 1.) These results suggest that a social planner should restrict use of big data to identify linkages over quality while encouraging use of big data to identify linkages over circumstances that are shared by small populations.

5 Data Sharing, Markets, and Consumer Welfare

So far we have considered the implications of data linkages for a single firm which uses data to inform predictions about consumer behavior. This focus allowed us to isolate the direct effect of data linkages on consumer effort and participation. When multiple firms compete for consumers, additional important questions regarding behavior and welfare arise which we can leverage our model to answer.

In this section we address a recent policy debate regarding data sharing. In many markets, a consumer's business brings with it data on the consumer's behavior, which by default is privately owned by the organization with which the consumer interacts. Recently, proposals have been made to form so-called "data commons" to make this data freely accessible to all organizations in the market. For example, the European Commission has begun exploring legislative action that would support "business-to-business data sharing," and new platforms for data sharing, such as Data Republic, permit organizations to share anonymised data with one another.[18] We study here the impact of such data sharing on effort provision and consumer welfare.[19]

To do this, we extend our model to $K \geq 2$ firms who compete over N consumers according to the following timeline:

$t = -1$: Each firm k simultaneously chooses a reward R_k. These transfers are publicly

[18] See `https://www.zdnet.com/article/data-republic-facilitates-diplomatic-data-sharing-on-aws/`.

[19] Our focus on consumer welfare mirrors recent policy discussions regarding data collection and sharing, which have been mostly concerned with the impact of these activities on consumers. Our main findings would be similar if we instead analyzed total social surplus. In particular, an analog of Proposition 3 holds when considering the impact of data sharing on social surplus.

observed.

$t = 0:$ Each consumer chooses a firm to participate with (if any).

$t = 1:$ Participating consumers choose what level of effort to exert, without observing the participation decisions of other consumers.

$t = 2:$ Participating consumers receive their firm's forecast of their type.

Payoffs and consumer welfare are as in the single-principal model.

We contrast a *proprietary data* regime, under which each firm observes only the outcomes of the consumers who interact with them, with a *data sharing* regime, under which the outcomes of all participating agents are shared across firms. These settings differ only in the information that firms have access to when making their forecasts at time $t = 2$. We assume that whether data is proprietary or shared is common knowledge.

As our solution concept, we use subgame-perfect Nash equilibria in pure strategies (which we henceforth refer to simply as an *equilibrium*).[20] Throughout, we maintain a restriction on out-of-equilibrium beliefs analogous to the refinement imposed in the single-principal model: at any information set in which agent i participates with principal k, principal k expects agent i to choose the action a_i satisfying

$$MV\left(1 + N_{-i}^k\right) = C'(a_i),$$

where N_{-i}^k is the number of agents $j \neq i$ who participate with principal k under their equilibrium strategies. This refinement ensures that each principal expects every participating agent i to choose the equilibrium action from a game with exogenous participation of $1 + N_{-i}^k$ agents, even when participation by agent i is out-of-equilibrium.

We do not provide a full characterization of the equilibrium set, as there exists a large set of equilibria under proprietary data.[21] Despite this fact, we can show that the shift from proprietary data to data sharing improves consumer welfare, no matter the equilibrium selection or the nature of linkages between consumers.

[20]This restriction differs slightly from the one we used in the single-principal model: we require that agents not mix over participation, but we allow agents to make asymmetric participation decisions. Imposing these restrictions in the single-principal model would not substantively impact the analysis. In particular, equilibria would be identical except in the circumstance linkage model with $N > N^*$. In that regime there exist pure-strategy equilibria with asymmetric entry decisions, which exhibit the same comparative statics in effort and participation rates as the symmetric mixed equilibrium.

[21]For a given set of transfers, strategic substitutibility or complementarity between consumer participation decisions allow for existence of a multiplicity of participation patterns. The selection of participation patterns across subgames can then support a variety of equilibrium rewards by firms.

Proposition 3. *In both the quality linkage and circumstance linkage models, consumer welfare is higher under data sharing than under proprietary data.*

This result arises from the interplay of two forces—how data sharing impacts the total surplus generated from the market via participation and effort, and how it changes the split of this surplus between consumers and firms. Under data sharing, firms are identical to consumers, since all firms have access to the same outcomes regardless of the pattern of participation. This forces firm profits to zero and transfers all surplus to consumers. On the other hand, data sharing has a potentially ambiguous impact on total surplus. Total surplus is rising in effort,[22] and effort is rising in the number of participating agents under circumstance linkages, but falling under quality linkages (Proposition 1). So while consumer welfare clearly rises in the circumstance linkage models, the result under quality linkages is more subtle.

We establish the result for the quality linkage model by proving that under proprietary data, in every equilibrium agents endogenously choose to interact with a single firm (Lemma E.2). This means that data sharing does *not* increase the effective population size, and aggregate surplus is the same with or without data sharing. The impact of data sharing on consumer welfare is then completely determined by the split of surplus, which we already observed is maximized for consumers under data sharing. So consumer welfare must be at least as large under this regime.

Proposition 3 indicates that under either kind of linkage across consumer outcomes, the introduction of data sharing is welfare-improving for consumers. This result does not imply that under data sharing, the identification of linkages always increases consumer welfare. As noted in Section 4.2, introduction of a quality linkage increases consumer welfare, but introduction of a circumstance linkage diminishes it. Thus, data sharing (the pooling of information across competitive firms) and data linkages (the identification of relationships among consumers that make one consumer's outcomes predictive of another's), while related, play very different roles: Data linkages determine how the size of a firm's consumer base impacts the effort that each consumer exerts; while data sharing determines the pattern of participation across the firms and how surplus is divided between consumers and firms. The results of this section reveal that data linkages and data sharing interact in important ways.

[22]More precisely, total surplus is rising in effort on the interval $[0, a_{FB}]$, where a_{FB} is the first-best action satisfying $C'(a_{FB}) = 1$. We showed in Lemma 2 that equilibrium actions are bounded below first-best. Thus, on the relevant domain, total surplus is rising in effort.

6 Extensions

6.1 Robustness to Uncertainty

We have so far supposed that consumers know the total population size N and the structure of correlation across the outcomes S_i. In practice, consumers may not have this kind of detailed knowledge about their segment. We show next that our qualitative findings remain unchanged when agents have uncertainty about the strength of correlation across outcomes and about the population size, so long as agents know whether consumers in their segment are related by quality or circumstance.

Formally, suppose that in the quality linkage model agents may be grouped into any of K "quality linkage" segments, each of which corresponds to a different correlation structure across types; that is, $\overline{\theta} \sim F^k_{\overline{\theta}}$, $\theta_i^\perp \sim F^k_{\theta^\perp}$, and $\varepsilon_i \sim F^k_\varepsilon$ for segment $k = 1, ..., K$. All agents share a common belief about the probability that they are in each segment. (The case of K "circumstance linkage" segments may be similarly defined.) At the same time, suppose that the number of agents N is a random variable, potentially dependent on the segment, with distribution $N \sim G^k_\gamma$, where γ is a scale factor known to all agents such that for each segment k, G^k_γ first-order stochastically dominates $G^k_{\gamma'}$ whenever $\gamma > \gamma'$.

Under this specification, the first-order condition characterizing optimal effort when agents enter with probability p may be written

$$\mathbb{E}\left[MV(1+\widetilde{N}, k)\right] = C'(a^*),$$

where $MV(N', k)$ is the marginal value of distortion when N' agents enter and the consumer is part of segment k, $\widetilde{N} \sim \text{Bin}(N-1, p)$, and N and k are both random variables. Note that for each segment k, $MV(N, k)$ changes with N just as in Lemma 1. Then conditional on the segment k, $\mathbb{E}[MV(1+\widetilde{N}, k) \mid k]$ decreases with p and γ in the quality linkage model, and increases with p and γ in the circumstance linkage model. Since this property holds for every segment k, it must also hold for the unconditional expected marginal value $\mathbb{E}\left[MV(1+\widetilde{N}, k)\right]$.

The reasoning of the previous paragraph yields the conclusion that the expected marginal value of distortion moves with the population scale factor γ and the entry rate p just as it does with respect to N and p in the baseline model. So the following corollary holds:

Corollary. *In the model with uncertainty over segment and population size, equilibrium effort and participation rates exhibit the same comparative statics in γ as with respect to N in Theorems 1 and 2.*

That is, an increase in γ—which shifts up the distribution for the number of participants no matter the realized segment—leads to higher effort under circumstance linkages and lower effort under quality linkage.[23]

6.2 Multiple Linkages

So far we have conducted our analysis supposing that each consumer is identified as part of a single segment. In practice a consumer may belong to several demographic and lifestyle segments, each of which may be used by an organization to improve predictions of the consumer's type. We now show that aggregation of outcomes from multiple segments for prediction creates a natural amplification of the effort effect identified in Proposition 1: as the number of identifiable quality linkages for a consumer increases (e.g. because the organization has purchased data about additional covariates), his effort declines; and as the number of identifiable circumstance linkages for a consumer increases, his effort rises.

To formally model variation in the number of segments, we focus on the effort exerted by a single agent, who we refer to as agent 0. We decompose the agent's outcome S_0 as the sum of a number of components, some common and some idiosyncratic. In the quality linkage context, we write

$$S_0 = a_0 + \sum_{j=1}^{J} \overline{\theta}^j + \theta_0^{\perp} + \varepsilon_0,$$

where $\theta_0^{\perp}$ and ε_0 are idiosyncratic persistent and transient components of the outcome. Each $\overline{\theta}^j$ is a persistent component of the outcome which is held in common with a segment j consisting of N_j agents. The outcomes of agents in segment j are observed by the principal, and each agent i in this segment has an outcome distributed as

$$S_i^j = a_i^j + \overline{\theta}^j + \theta_i^{\perp,j} + \varepsilon_i^j$$

where $\theta_i^{\perp,j}$ and ε_i^j are idiosyncratic.[24] As usual, the principal wishes to predict $\theta_0 = \sum_{j=1}^{J} \overline{\theta}^j + \theta_i^{\perp}$. Analogously, in the circumstance linkage model we decompose the agent's outcome as

$$S_0 = a_0 + \theta_0 + \sum_{j=1}^{J} \overline{\varepsilon}^j + \varepsilon_0^{\perp},$$

[23]The threshold N^* at which participation rates begin to drop in the "circumstance linkage" case would, however, depend on details of their beliefs about the segment.

[24]For simplicity, we do not model agents in other groups as having multiple linkages. Extending the model to allow such linkages would not impact results in any way so long as no group j is linked to another group j' also linked to agent 0.

where each agent i from group j has an outcome distributed as

$$S_i^j = a_i^j + \theta_i^j + \overline{\varepsilon}^j + \varepsilon_i^{\perp,j}.$$

As in the baseline model, all type and shock terms are mutually independent. In each model we impose analogs of the assumptions in Section 2.6 on the relevant densities and posterior means. Participation of all agents is exogenously given.

Proposition 4 below demonstrates a comparative static in the number of linkages observed by the principal. A principal who observes m linkages understands the correlation structure of each $\overline{\theta}^j$ (or $\overline{\varepsilon}^j$) with the segment-j outcomes $(S_1^j, ..., S_{N_j}^j)$ for $j = 1, ..., m$, but believes that for $j = m+1, ..., J$ each $\overline{\theta}^j$ (or $\overline{\varepsilon}^j$) term is idiosyncratic. This could, for example, correspond to the principal knowing which of their consumers are charitable givers, but not knowing which consumers are single parents. Let $a_Q^\dagger(m)$ be agent 0's equilibrium action when the principal observes m linkages in the quality linkage model, with $a_C^\dagger(m)$ similarly defined for the circumstance linkage model. The following result characterizes how agent 0's equilibrium action changes with m.

Proposition 4. *$a_Q^\dagger(m)$ is strictly decreasing in m, while $a_C^\dagger(m)$ is strictly increasing in m.*

For simplicity we have restricted attention to multiple linkages of the same type. However, the basic logic of Proposition 4 holds even when the agent may be linked to other segments via both quality and circumstance linkages. Given any initial set of linkages (each of which may be either a quality or circumstance linkage), identification of an additional quality linkage decreases equilibrium effort, while identification of an additional circumstance linkage increases equilibrium effort. (We omit the proof, which follows straightforwardly along the lines of the proof of Proposition 4.)

6.3 Forecast Prediction and Welfare

So far we have considered prediction of an agent's type relevant for social welfare only insofar as it generates incentives for the agent to exert effort to influence the prediction. However, in some applications, better tailoring of a service level to fit the agent's type may involve changes in allocation which improve welfare. For instance, a bank extending loans to small businesses may increase total output if it is able to more accurately match loan amounts to the profitability of each business.

When better prediction improves welfare, the social welfare results of Proposition 2 are qualitatively the same for circumstance linkages, but may change under a quality linkage. Identification of a circumstance linkage now has two positive forces on per-agent welfare, improving both the effort exerted and the forecast precision of each participating agent's type (given a fixed entry rate). Since the participation rate still drops to zero when the population size becomes large, circumstance linkages improve welfare for small populations but decrease it for large populations, identical to the baseline model.

Under a quality linkage, the impact of the linkage on effort and prediction accuracy have countervailing effects on welfare. For large populations the total effect is determined by the comparison between drop of effort from $a^*(1)$ to $\lim_{N\to\infty} a^*(N)$ versus the gains from accurate prediction of $\overline{\theta}$. When the value to improved prediction is small, quality linkages decrease welfare for large populations (as in our baseline model), while the opposite is true when the value to improved prediction is large.

7 Conclusion

As firms and governments move towards collecting large datasets of consumer transactions and behavior as inputs to decision-making, the question of whether and how to regulate the usage of consumer data has emerged as an important policy question. Recent regulations, such as the European Union's General Data Protection Regulation (GDPR), have focused on protecting consumers' privacy and improving transparency regarding what kind of data is being collected. An important complementary consideration when designing regulations is how data impacts social and economic behaviors.

In the present paper, we analyze one such impact—the effect that consumer segmentations identified by novel datasets have on consumer incentives for socially valuable effort. We find that the behavioral and welfare consequences depend crucially on how consumers in a segment are linked. These results suggest that regulations should take into account not just whether individual data is informative about other consumers, but whether that data is primarily useful for inferring quality or denoising observations.

In practice, the usage of a particular dataset is likely to differ across domains, and may have as much to do with the underlying correlation structure of the data as it does with the algorithms used to aggregate that data. We hope that even the reduced-form models of data aggregation that we have considered here make clear that regulation of the "amount" of data is too crude for many objectives—the structure of that data, and how it is used for prediction, can have important consequences.

Finally, our analysis in Section 5 of the interaction between market forces and data linkages points to another interesting avenue for subsequent work. Since participation is a strategic complement under quality linkages but a strategic substitute under circumstance linkages, the former encourages the emergence of a single firm that serves all consumers, while the latter discourages it. This suggests that identification of linkages across consumers affects not just those consumers and their behavior, but can also have important implications for market structure and antitrust policy.

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Appendix

The appendices are structured as follows: Appendix A reports a list of actual consumer data segmentations sold by data brokers. Appendix B establishes technical results used in the proofs of the results in the body of the paper. The remaining appendices present proofs of all results in the body of the paper.

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A Consumer Segments Provided by Data Brokers

In this appendix we produce a list of examples of actual consumer segmentations produced by data brokers, as reported in Federal Trade Commission (2014) and Senate Committee on Commerce, Science, and Transportation (2013).

Table 1: Examples of Consumer Segments

Quality Linkage	**Circumstance Linkage**
Outdoor/Hunting & Shooting	Sending a Kid to College
Santa Fe/Native American Lifestyle	Expectant Parents
Media Channel Usage - Daytime TV	Buying a Home
Bible Lifestyle	Getting Married
New Age/Organic Lifestyle	Dieters
Plus-size Apparel	Families with Newborns
Biker/Hell's Angels	Hard Times
Leans Left	New Mover/Renter/Owner
Fitness Enthusiast	Death in the Family
Working-class Mom	
Thrifty Elders	
Health & Wellness Interest	
Very Spartan	
Small Town Shallow Pockets	
Established Elite	
Frugal Families	
McMansions & Minivans	

We have informally categorized segments according to whether they might represent a quality linkage or a circumstance linkage; in practice, this categorization would depend also on the time frame for forecasting. For example, a segment of "consumers with children in college" during a particular observation cycle is a quality linkage segment while the children remain in college, but a circumstance linkage segment once the children have graduated.

Besides these named categories, data brokers provide also segmentation based on numerous demographic, health, interest, financial, and social media indicators, including: miles traveled in the last 4 weeks, number of whiskey drinks consumed in the past 30 days, whether the individual or household is a pet owner, whether the individual donates to charitable causes, whether the individual enjoys reading romance novels, whether the individual participates in sweepstakes or contests, whether the individual suffers from allergies, whether the individual is a member of five or more social networks, whether individual is a heavy Twitter user, among countless others.

B Preliminary Results

In this section we establish a number of first-order stochastic dominance and monotonicity results used in proofs of results in the body of the paper. Throughout this appendix, fix a segment size N and assume that all agents opt in. (All results extend immediately to any set of agents $I \subset \{1, ..., N'\}$ of size N entering from a segment of size $N' > N$.) Let G_i^M denote the distribution function of agent i's outcome in model $M \in \{Q, C\}$, with $M = Q$ the quality linkage model and $M = C$ the circumstance linkage model. We will write g_i^M for the density function associated with G_i^M. For the joint distribution of the outcomes of agents i through j, we will write $G_{i:j}^M$,

B.1 Smooth MLRP

A classic result of Milgrom (1981) demonstrates that if a signal satisfies the monotone likelihood ratio property (MLRP), then posterior beliefs can be ordered by first-order stochastic dominance. For our results we desire not just that the posterior distribution is strictly decreasing in the conditioning variable, but that it be differentiable and that the derivative be strictly negative. We define a smooth form of the MLRP sufficient to achieve this result.

Definition B.1 (Smooth MLRP). *A family of conditional density functions* $\{f(x \mid y)\}_{y \in Y}$ *on* $\mathbb{R}$ *for some* $Y \subset \mathbb{R}$ *satisfies the* smooth monotone likelihood ratio property (SMLRP) *in* y *if:*

- $f(x \mid y)$ *is a strictly positive,* $C^{1,0}$ *function*[25] *of* (x, y),
- $f(x \mid y)$ *and* $\frac{\partial}{\partial x} f(x \mid y)$ *are both uniformly bounded for all* (x, y),
- *The likelihood ratio function*

$$\ell(x; y, y') \equiv \frac{f(x \mid y)}{f(x \mid y')}$$

satisfies $\frac{\partial \ell}{\partial x}(x; y, y') > 0$ *for every* x *and* $y > y'$.

[25] A function $f : \mathbb{R}^2 \to \mathbb{R}$ lies in the class $C^{1,0}$ if it is continuous everywhere and $\frac{\partial f}{\partial x}(x, y)$ exists and is continuous everywhere.

This definition is a strengthening of the MLRP definition of Milgrom (1981). It requires not only that the likelihood ratio function be everywhere strictly increasing, but that it be differentiable with the derivative strictly positive. It also imposes regularity conditions on the likelihood and its derivative which will be necessary for the desired FOSD result to hold.

One useful identity involving the likelihood ratio function is

$$\frac{\partial \ell}{\partial x}(x;y,y') = \frac{f(x|y)}{f(x|y')}\left(\frac{\partial}{\partial x}\log f(x|y) - \frac{\partial}{\partial x}\log f(x|y')\right).$$

Thus the condition on the likelihood ratio function imposed by SMLRP is equivalent to the condition that $\frac{\partial}{\partial x}\log f(x|y)$ be a strictly increasing function of y for every x.

The following lemma establishes a very important class of random variables satisfying SMLRP.

Lemma B.1. *Let X and Y be two independent random variables with density functions f_X and f_Y which are each C^1, strictly positive, strictly log-concave functions, and which each have bounded first derivative. Let $Z = k + X + Y$ for a constant k. Then the conditional densities $f_{Z|X}(z \mid x)$ and $f_{Z|Y}(z \mid y)$ satisfy the SMLRP in x and y, respectively.*

Proof. First take $k = 0$. We prove the result for $f_{Z|X}$, with the result for $f_{Z|Y}$ following symmetrically. Note that $f_{Z|X}(z \mid x) = f_Y(z - x)$. By Lemma O.1, f_Y is bounded. This result along with the additional assumptions on f_Y ensure that $f_{Z|X}$ satisfies the first two conditions of SMLRP. As for the likelihood ratio condition, it is sufficient to establish that $\frac{\partial}{\partial z}\log f_{Z|X}(z \mid x) = \frac{\partial}{\partial z}\log f_Y(z-x)$ is strictly increasing in x for each z. But since f_Y is strictly log-concave, $\frac{\partial}{\partial z}\log f_Y(z-x) > \frac{\partial}{\partial z}\log f_Y(z-x')$ whenever $z-x < z-x'$, i.e. whenever $x > x'$. So the likelihood ratio condition is satisfied as well.

Now suppose $k \neq 0$. Then the result applied to the random variable $X + Y$ establishes that $f_{X+Y|X}(z \mid x)$ and $f_{X+Y|Y}(z \mid y)$ satisfy the SMLRP in x and y, respectively. As $f_{Z|X}(z \mid x) = f_{X+Y|X}(z-k \mid x)$ and $f_{Z|Y}(z \mid y) = f_{X+Y|Y}(z-k \mid y)$, and since each of the conditions of the SMLRP are invariant to shifts in the first argument, these densities satisfy the SMLRP as well. □

The following lemma is the main result of this appendix. It strengthens the FOSD result of Milgrom (1981) to ensure that the posterior distribution function is smooth and has a strictly negative derivative wrt the conditioning variable. The sufficient conditions are that the likelihood function satisfy SMLRP and that the density function of the unobserved variable be continuous. The proof here establishes the sign of the derivative, with the proof of smoothness relegated to Lemma O.2 in the Online Appendix.

Lemma B.2 (Smooth FOSD). *Let X and Y be two random variables for which the density $g(y)$ for Y and the conditional densities $f(x \mid y)$ for $X \mid Y$ exist. Suppose that $f(x \mid y)$ satisfies the SMLRP in y and $g(y)$ is continuous. Then $H(x, y) \equiv \Pr(Y \leq y \mid X = x)$ is a C^1 function of (x, y) and $\frac{\partial H}{\partial x}(x, y) < 0$ everywhere.*

Proof. Lemma O.2 establishes that H is a C^1 function. To sign its derivative wrt x, note that the derivative of $\widehat{H}(x, y) \equiv H(x, y)^{-1} - 1$ may manipulated to obtain the form

$$\frac{\partial \widehat{H}}{\partial x}(x, y) = \left(\int_{-\infty}^{y} f(x \mid y'')\, dG(y'')\right)^{-2}$$
$$\times \int_{y}^{\infty} dG(y') \int_{-\infty}^{y} dG(y'') \left(f(x \mid y'') \frac{\partial}{\partial x} f(x \mid y') - f(x \mid y') \frac{\partial}{\partial x} f(x \mid y'')\right).$$

(See the proof of Lemma O.2 for a detailed derivation.) The integrand may be rewritten

$$\begin{aligned} &f(x \mid y'') \frac{\partial}{\partial x} f(x \mid y') - f(x \mid y') \frac{\partial}{\partial x} f(x \mid y'') \\ &= f(x \mid y'')^2 \left(\frac{\frac{\partial}{\partial x} f(x \mid y')}{f(x \mid y'')} - \frac{f(x \mid y') \frac{\partial}{\partial x} f(x \mid y'')}{f(x \mid y'')^2} \right) \\ &= f(x \mid y'')^2 \frac{\partial}{\partial x} \ell(x; y', y''). \end{aligned}$$

Now, as $y' > y > y''$ on the interior of the domain of integration, $\frac{\partial}{\partial x} \ell(x; y', y'') > 0$ everywhere and so $\frac{\partial \widehat{H}}{\partial x}(x, y) > 0$. Therefore

$$\frac{\partial H}{\partial x}(x, y) = -\frac{\frac{\partial \widehat{H}}{\partial x}(x, y)}{(\widehat{H}(x, y) + 1)^2} < 0,$$

as desired. □

B.2 SFOSD of Posterior Distributions

We now develop smooth first-order stochastic dominance results regarding posterior distributions of various latent variables as outcomes shift. These results rely heavily on the SFOSD result established in Lemma B.2. Application of that lemma requires checking smoothness and boundedness conditions of the underlying likelihood functions, which are straightforward but tedious in our environment. We relegate proofs of these regularity conditions to Online Appendix O.1.

The following result establishes that as an agent's outcome increases, inferences about the common component of the outcome increase as well.

Lemma B.3. *For agent $i \in \{1, ..., N\}$ and outcome-action profile $(\mathbf{S}_{-i}, \mathbf{a})$:*

- $F^Q_{\overline{\theta}}(\overline{\theta} \mid \mathbf{S};\mathbf{a})$ *is a* C^1 *function of* $(S_i, \overline{\theta})$ *satisfying* $\frac{\partial}{\partial S_i} F^Q_{\overline{\theta}}(\overline{\theta} \mid \mathbf{S};\mathbf{a}) < 0$ *for all* $(S_i, \overline{\theta})$,
- $F^C_{\overline{\varepsilon}}(\overline{\varepsilon} \mid \mathbf{S};\mathbf{a})$ *is a* C^1 *function of* $(S_i, \overline{\varepsilon})$ *satisfying* $\frac{\partial}{\partial S_i} F^C_{\overline{\varepsilon}}(\overline{\varepsilon} \mid \mathbf{S};\mathbf{a}) < 0$ *for all* $(S_i, \overline{\varepsilon})$,

Proof. For convenience, we suppress the dependence of distributions on $\mathbf{a}$ in this proof. Fix $\mathbf{S}_{-i}$. We will prove the first result, with the second following from nearly identical work by permuting the roles of θ and ε.

The result follows from Lemma B.2 provided that 1) $f^Q_{\overline{\theta}}(\overline{\theta} \mid \mathbf{S}_{-i})$ is continuous wrt $\overline{\theta}$, and 2) $g^Q_i(S_i \mid \overline{\theta}, \mathbf{S}_{-i})$ satisfies SMLRP with respect to $\overline{\theta}$. As for the first condition, Bayes' rule gives

$$f^Q_{\overline{\theta}}(\overline{\theta} \mid \mathbf{S}_{-i}) = \frac{f_{\overline{\theta}}(\overline{\theta}) \prod_{j \neq i} g_j(S_j \mid \overline{\theta})}{g_{-i}(\overline{S}_{-i})} = \frac{f_{\overline{\theta}}(\overline{\theta}) \prod_{j \neq i} f_{\theta^\perp + \varepsilon}(S_j - \overline{\theta} - a_j)}{g_{-i}(\overline{S}_{-i})}.$$

Then as $f_{\overline{\theta}}$ and $f_{\theta^\perp+\varepsilon}$ are both continuous functions, $f^Q_{\overline{\theta}}(\overline{\theta} \mid \mathbf{S}_{-i})$ is a continuous function of $\overline{\theta}$. It therefore suffices to establish condition 2.

Note that conditional on $\overline{\theta}$, S_i is independent of $\mathbf{S}_{-i}$ in the quality linkage model; so $g^Q_i(S_i \mid \overline{\theta}, \mathbf{S}_{-i}) = g^Q_i(S_i \mid \overline{\theta})$. So it suffices to establish that $g^Q_i(S_i \mid \overline{\theta})$ satisfies SMLRP with respect to $\overline{\theta}$. Recall that in the quality linkage model, $S_i = a_i + \overline{\theta} + \theta^\perp_i + \varepsilon_i$, where by assumption $\overline{\theta}$, $\theta^\perp_i$ and ε_i all have C^1, strictly positive, strictly log-concave density functions with bounded derivatives. Lemma O.1 ensures that these densities are additionally bounded. These properties are all inherited by the density function of the sum $\theta^\perp_i + \varepsilon_i$, which is just the convolution of the density functions for $\theta^\perp_i$ and ε_i. Lemma B.1 then implies that $g^Q_i(S_i \mid \overline{\theta})$ satisfies SMLRP with respect to $\overline{\theta}$, as desired. □

The following lemma establishes smooth stochastic dominance of a posterior distribution arising in analysis of the quality linkage model. While the property is the same one established by Lemma B.2, the boundedness conditions of that lemma cannot be guaranteed and so slightly different techniques are required to reach the result.

Lemma B.4. *For every outcome-action profile* (S_1, a_1) *and type* θ_1*, the function* $F^Q_{\theta_1}(\theta_1 \mid S_1, \overline{\theta}; a_1)$ *is continuously differentiable wrt* $\overline{\theta}$ *everywhere, and* $\frac{\partial}{\partial \overline{\theta}} F^Q_{\theta_1}(\theta_1 \mid S_1, \overline{\theta}; a_1) < 0$.

Proof. For convenience, we suppress the dependence of distributions on a_1 in this proof. By Bayes' rule,

$$F^Q_{\theta_1}(t \mid S_1, \overline{\theta}) = \frac{\int_{-\infty}^{t} f_{\overline{\theta}}(\overline{\theta} \mid \theta_1 = t', S_1) f_{\theta_1}(t' \mid S_1)\, dt'}{\int_{-\infty}^{\infty} f_{\overline{\theta}}(\overline{\theta} \mid \theta_1 = t', S_1) f_{\theta_1}(t' \mid S_1)\, dt'}.$$

Note that $f^Q_{\overline{\theta}}(\overline{\theta} \mid \theta_1, S_1)$ is independent of S_1, as (θ_1, S_1) contains the same information as $(\theta_1, \varepsilon_1)$ and $\overline{\theta}$ is independent of ε_1. So $f^Q_{\overline{\theta}}(\overline{\theta} \mid \theta_1, S_1) = f^Q_{\overline{\theta}}(\overline{\theta} \mid \theta_1)$. Another application of

Bayes' rule reveals that

$$f^Q_{\overline{\theta}}(\overline{\theta} \mid \theta_1) = \frac{f^Q_{\theta_1}(\theta_1 \mid \overline{\theta}) f_{\overline{\theta}}(\overline{\theta})}{f_\theta(\theta_1)} = \frac{f_{\theta^\perp}(\theta_1 - \overline{\theta}) f_{\overline{\theta}}(\overline{\theta})}{f_\theta(\theta_1)},$$

while

$$f_{\theta_1}(\theta_1 \mid S_1) = \frac{g_1(S_1 \mid \theta_1) f_\theta(\theta_1)}{g(S_1)} = \frac{f_\varepsilon(S_1 - \theta_1 - a_1) f_\theta(\theta_1)}{g(S_1)}.$$

Inserting back into the previous expression for $F^Q_{\theta_1 = t}(\theta_1 \mid S_1, \overline{\theta})$ yields

$$F^Q_{\theta_1}(t \mid S_1, \overline{\theta}) = \frac{\int_{-\infty}^{t} f_{\theta^\perp}(t' - \overline{\theta}) f_\varepsilon(S_1 - t' - a_1)\, dt'}{\int_{-\infty}^{\infty} f_{\theta^\perp}(t' - \overline{\theta}) f_\varepsilon(S_1 - t' - a_1)\, dt'}.$$

Using the change of variables $t'' = S_1 - t' - a_1$ yields

$$F^Q_{\theta_1}(t \mid S_1, \overline{\theta}) = \frac{\int_{S_1 - t - a_1}^{\infty} f_{\theta^\perp}(S_1 - a_1 - \overline{\theta} - t'') dF_\varepsilon(t'')}{\int_{-\infty}^{\infty} f_{\theta^\perp}(S_1 - a_1 - \overline{\theta} - t'') dF_\varepsilon(t'')}.$$

Now, as $f'_{\theta^\perp}$ exists and is bounded, the Leibniz integral rule ensures that derivatives of the numerator and denominator wrt $\overline{\theta}$ may be moved inside the integral sign. So $F^Q_{\theta_1}(t \mid S_1, \overline{\theta})$ is differentiable wrt $\overline{\theta}$. And as $f'_{\theta^\perp}$ is additionally continuous, the dominated convergence theorem ensures that these derivatives are continuous. Meanwhile the numerator and denominator themselves are each continuous in $\overline{\theta}$ given that $f_{\theta^\perp}$ is continuous and bounded. Thus $F^Q_{\theta_1}(\theta_1 \mid S_1, \overline{\theta})$ is continuously differentiable wrt $\overline{\theta}$.

To sign the derivative, we may equivalently sign

$$H(\overline{\theta}) \equiv F^Q_{\theta_1}(t \mid S_1, \overline{\theta})^{-1} - 1 = \frac{\int_{-\infty}^{S_1 - t - a_1} f_{\theta^\perp}(S_1 - a_1 - \overline{\theta} - t') dF_\varepsilon(t')}{\int_{S_1 - t - a_1}^{\infty} f_{\theta^\perp}(S_1 - a_1 - \overline{\theta} - t'') dF_\varepsilon(t'')}.$$

Differentiating and re-arranging yields

$$\begin{aligned} H'(\overline{\theta}) = & \left(\int_{S_1 - t - a_1}^{\infty} f_{\theta^\perp}(S_1 - a_1 - \overline{\theta} - t'') dF_\varepsilon(t'') \right)^{-2} \\ & \times \int_{-\infty}^{S_1 - t - a_1} dF_\varepsilon(t') \int_{S_1 - t - a_1}^{\infty} dF_\varepsilon(t'') \\ & \qquad \times \big(-f_{\theta^\perp}(S_1 - a_1 - \overline{\theta} - t'') f'_{\theta^\perp}(S_1 - a_1 - \overline{\theta} - t') \\ & \qquad\quad + f_{\theta^\perp}(S_1 - a_1 - \overline{\theta} - t') f'_{\theta^\perp}(S_1 - a_1 - \overline{\theta} - t'') \big). \end{aligned}$$

The integrand may be rewritten

$$\begin{aligned} & - f_{\theta^\perp}(S_1 - a_1 - \overline{\theta} - t'') f'_{\theta^\perp}(S_1 - a_1 - \overline{\theta} - t') \\ & + f_{\theta^\perp}(S_1 - a_1 - \overline{\theta} - t') f'_{\theta^\perp}(S_1 - a_1 - \overline{\theta} - t'') \\ = & \, f_{\theta^\perp}(S_1 - a_1 - \overline{\theta} - t'') f_{\theta^\perp}(S_1 - a_1 - \overline{\theta} - t') \\ & \times \left(- \frac{f'_{\theta^\perp}(S_1 - a_1 - \overline{\theta} - t')}{f_{\theta^\perp}(S_1 - a_1 - \overline{\theta} - t')} + \frac{f'_{\theta^\perp}(S_1 - a_1 - \overline{\theta} - t'')}{f_{\theta^\perp}(S_1 - a_1 - \overline{\theta} - t'')} \right). \end{aligned}$$

Note that everywhere on the domain of integration $t'' > t'$, and so because $f_{\theta^\perp}$ is strictly log-concave,

$$\frac{f'_{\theta^\perp}(S_1 - a_1 - \overline{\theta} - t'')}{f_{\theta^\perp}(S_1 - a_1 - \overline{\theta} - t'')} > \frac{f'_{\theta^\perp}(S_1 - a_1 - \overline{\theta} - t')}{f_{\theta^\perp}(S_1 - a_1 - \overline{\theta} - t')}.$$

Thus the integrand is strictly positive everywhere, meaning $H'(\overline{\theta}) > 0$. In other words,

$$\frac{\partial}{\partial \overline{\theta}} F^Q_{\theta_1}(\theta_1 \mid S_1, \overline{\theta}) = -\frac{H'(\overline{\theta})}{(H(\overline{\theta}) + 1)^2} < 0,$$

as desired. □

The following lemma establishes how inferences about one agent's quality change as another agent's outcome changes. Note that the result depends critically on the model. For simplicity, the result is stated in terms of inferences about agent 1's type as agent N's outcome shifts. By symmetry analogous results hold for any other pair of agents.

Lemma B.5. *For every outcome-action profile* $(\mathbf{S}_{-N}, \mathbf{a})$,

$$\frac{\partial}{\partial S_N} F^Q_{\theta_1}(\theta_1 \mid \mathbf{S}; \mathbf{a}) < 0$$

and

$$\frac{\partial}{\partial S_N} F^C_{\theta_1}(\theta_1 \mid \mathbf{S}; \mathbf{a}) > 0$$

for every (θ_1, S_N).

Proof. For convenience, we suppress the dependence of distributions on $\mathbf{a}$ in this proof. Fix $\mathbf{S}_{-N}$. Recall that Lemma O.4 established that $F^M_{\theta_1}(\theta_1 \mid \mathbf{S})$ is a C^1 function of (S_N, θ_1) for each model $M \in \{Q, C\}$.

Consider first the quality linkage model. Then

$$F^Q_{\theta_1}(\theta_1 \mid \mathbf{S}) = \int_{-\infty}^{\infty} F^Q_{\theta_1}(\theta_1 \mid \mathbf{S}, \overline{\theta})\, dF^Q_{\overline{\theta}}(\overline{\theta} \mid \mathbf{S}).$$

Conditional on $\overline{\theta}$, θ_1 depends on $\mathbf{S}$ only through S_1, so this can be written

$$F^Q_{\theta_1}(\theta_1 \mid \mathbf{S}) = \int_{-\infty}^{\infty} F^Q_{\theta_1}(\theta_1 \mid S_1, \overline{\theta})\, dF^Q_{\overline{\theta}}(\overline{\theta} \mid \mathbf{S}).$$

Lemma B.3 establishes that $F^Q_{\overline{\theta}}(\overline{\theta} \mid \mathbf{S})$ is a C^1 function of $(S_N, \overline{\theta})$ satisfying $\frac{\partial}{\partial S_N} F^Q_{\overline{\theta}}(\overline{\theta} \mid \mathbf{S}) < 0$ everywhere. Then the function $F^Q_{\overline{\theta}}(\overline{\theta} \mid \mathbf{S}) - q$ is a C^1 function of $(S_N, \overline{\theta}, q)$, with Jacobian $f^Q_{\overline{\theta}}(\overline{\theta} \mid \mathbf{S})$ wrt $\overline{\theta}$. By Bayes' rule,

$$f^Q_{\overline{\theta}}(\overline{\theta} \mid \mathbf{S}) = \frac{f_{\overline{\theta}}(\overline{\theta}) \prod_{i=1}^N g_i(S_i \mid \overline{\theta})}{\int d\overline{\theta}'\, f_{\overline{\theta}}(\overline{\theta}') \prod_{i=1}^N g_i(S_i \mid \overline{\theta}')}.$$

As $g_i(S_i \mid \overline{\theta}) = f_{\theta^\perp+\varepsilon}(S_i - \overline{\theta} - a_i)$ and $f_{\overline{\theta}}$ and $f_{\theta^\perp+\varepsilon}$ are both strictly positive, $f^Q_{\overline{\theta}}(\overline{\theta} \mid \mathbf{S}) > 0$ everywhere. Therefore by the implicit function theorem there exists a C^1 function $\phi(q, S_N)$ such that $F^Q_{\overline{\theta}}(\phi(q, S_N) \mid \mathbf{S}) = q$ for all (q, S_N), and further that

$$\frac{\partial \phi}{\partial S_N}(q, S_N) = -\left[\frac{1}{f^Q_{\overline{\theta}}(t \mid \mathbf{S})} \frac{\partial}{\partial S_N} F^Q_{\overline{\theta}}(t \mid \mathbf{S})\right]_{t=\phi(q,S_N)} > 0.$$

A change of variables allows $F^Q_{\theta_1}(\theta_1 \mid \mathbf{S})$ to be integrated with respect to quantiles of $\overline{\theta}$ using the quantile function ϕ, yielding

$$F^Q_{\theta_1}(\theta_1 \mid \mathbf{S}) = \int_0^1 F^Q_{\theta_1}(\theta_1 \mid S_1, \overline{\theta} = \phi(q, S_N))\, dq.$$

Then for any $\Delta > 0$,

$$\begin{aligned} &-\frac{1}{\Delta}\left(F^Q_{\theta_1}(\theta_1 \mid S_N = s_N + \Delta, \mathbf{S}_{-N}) - F^Q_{\theta_1}(\theta_1 \mid S_N = s_N, \mathbf{S}_{-1})\right) \\ &= \int_0^1 -\frac{1}{\Delta}\left(F^Q_{\theta_1}(\theta_1 \mid S_1, \overline{\theta} = \phi(q, s_N + \Delta)) - F^Q_{\theta_1}(\theta_1 \mid S_1, \overline{\theta} = \phi(q, s_N))\right) dq. \end{aligned}$$

Since $F^Q_{\theta_1}(\theta_1 \mid \mathbf{S})$ is differentiable wrt S_N, the limit of both sides as $\Delta \downarrow 0$ must be well-defined. Lemma B.4 establishes that $\frac{\partial}{\partial \overline{\theta}} F^Q_{\theta_1}(\theta_1 \mid S_1, \overline{\theta})$ exists, is continuous in $\overline{\theta}$, and is strictly negative everywhere. Meanwhile we showed above that $\phi(q, S_N)$ is strictly increasing in S_N. This means that the interior of the integrand is strictly positive for every q and $\Delta > 0$, implying by Fatou's lemma and the chain rule that

$$-\frac{\partial}{\partial S_N} F^Q_{\theta_1}(\theta_1 \mid \mathbf{S}) \geq -\int_0^1 \frac{\partial}{\partial \overline{\theta}} F^Q_{\theta_1}(\theta_1 \mid S_1, \overline{\theta})\bigg|_{\overline{\theta}=\phi(q,S_N)} \frac{\partial \phi}{\partial S_N}(q, S_N)\, dq.$$

As the first term in the integrand is strictly negative while the second is strictly positive, this inequality in turn implies

$$\frac{\partial}{\partial S_N} F^Q_{\theta_1}(\theta_1 \mid \mathbf{S}) < 0.$$

Now consider the circumstance linkage model. Virtually all of the work for the quality linkage model goes through with $\overline{\varepsilon}$ exchanged for $\overline{\theta}$, with the key exception that the existence, continuity, and sign of $\frac{\partial}{\partial \overline{\varepsilon}} F^C_{\theta_1}(\theta_1 \mid S_1, \overline{\varepsilon})$ must be established separately. (Lemma B.4 applies only to the quality linkage model.) Note that $F^C_{\theta_1}(\theta_1 \mid S_1 = s, \overline{\varepsilon} = t) = F^C_{\theta_1}(\theta_1 \mid \widetilde{S}_1 = s - t)$, where $\widetilde{S}_1 \equiv a_1 + \theta_1 + \varepsilon_1^\perp$. It is therefore sufficient to analyze $\frac{\partial}{\partial \widetilde{S}_1} F^C_{\theta_1}(\theta_1 \mid \widetilde{S}_1)$. Let $\widetilde{g}_1(\widetilde{S}_1 \mid \theta_1)$ be the density function of $\widetilde{S}_1$ conditional on θ_1. We invoke Lemma B.1 to conclude that $\widetilde{g}_1(\widetilde{S}_1 \mid \theta_1)$ satisfies SMLRP in θ_1. As additionally $f_\theta(\theta_1)$ is continuous by assumption, Lemma B.2 ensures that $\frac{\partial}{\partial \widetilde{S}_1} F^C_{\theta_1}(\theta_1 \mid \widetilde{S}_1)$ exists, is continuous, and is strictly negative everywhere. Thus $\frac{\partial}{\partial \overline{\varepsilon}} F^C_{\theta_1}(\theta_1 \mid S_1, \overline{\varepsilon})$ exists, is continuous, and is strictly positive everywhere.

In light of this result, the final steps of the proof from the quality linkage case adapted to the circumstance linkages model show that

$$\begin{aligned}
&\frac{1}{\Delta}\left(F^C_{\theta_1}(\theta_1 \mid S_N = s_N + \Delta, \mathbf{S}_{-N}) - F^C_{\theta_1}(\theta_1 \mid S_N = s_N, \mathbf{S}_{-1})\right) \\
= &\int_0^1 \frac{1}{\Delta}\left(F^C_{\theta_1}(\theta_1 \mid S_1, \overline{\varepsilon} = \phi(q, s_N + \Delta)) - F^C_{\theta_1}(\theta_1 \mid S_1, \overline{\varepsilon} = \phi(q, s_N))\right) dq,
\end{aligned}$$

where the interior of the right-hand side is strictly positive for all $\Delta > 0$. Then by Fatou's lemma and the chain rule

$$\frac{\partial}{\partial S_N} F^C_{\theta_1}(\theta_1 \mid \mathbf{S}) \geq \int_0^1 \frac{\partial}{\partial \overline{\varepsilon}} F^Q_{\theta_1}(\theta_1 \mid S_1, \overline{\varepsilon})\bigg|_{\overline{\varepsilon}=\phi(q,S_N)} \frac{\partial \phi}{\partial S_N}(q, S_N)\, dq > 0.$$

□

B.3 Monotonicity of Posterior Expectations

This appendix establishes a series of monotonicity results about how posterior expectations of various latent variables change as some agent's outcome shifts. These results are consequences of the SFOSD results derived in Appendix B.2. Several of the results require smoothness or positivity conditions on underlying distribution and density functions, which are straightforward but tedious to check in our environment. We relegate proofs of these properties to Online Appendix O.1.

We first establish that the posterior expectation of an agent's type increases in his own signal, and that the rate of increase is bounded strictly between 0 and 1.

Lemma B.6 (Forecast sensitivity)**.** *For each agent $i \in \{1, ..., N\}$ and outcome-action profile* $(\mathbf{S}, \mathbf{a})$,

$$0 < \frac{\partial}{\partial S_i} \mathbb{E}[\theta_i \mid \mathbf{S}; \mathbf{a}] < 1.$$

Proof. For convenience, we suppress the dependence of distributions on $\mathbf{a}$ throughout this proof. Also wlog consider agent $i = 1$. We establish the result for the quality linkage model, with the result for the circumstance linkage model following by nearly identical work.

Fix a vector of signal realizations $\mathbf{S}_{-1}$. First note that $g^Q_1(S_1 \mid \theta_1, \mathbf{S}_{-1}) = g^Q_1(S_1 \mid \theta_1)$, and S_1 is the sum of a constant plus the independent random variables θ_1 and ε_1, each of which has a C^1, strictly positive, strictly log-concave density function with bounded derivative. Thus by Lemma B.1 $g^Q_1(S_1 \mid \theta_1, \mathbf{S}_{-1})$ satisfies SMLRP with respect to θ_1. Further, $f^Q_{\theta_1}(\theta_1 \mid \mathbf{S}_{-1})$ is continuous in θ_1 by Lemma O.3. Lemma B.2 then ensures that $F^Q_{\theta_1}(\theta_1 \mid \mathbf{S})$ is a C^1 function of (θ_1, S_1) and $\frac{\partial}{\partial S_1} F^Q_{\theta_1}(\theta_1 \mid \mathbf{S}) < 0$ everywhere.

Meanwhile conditional on $\mathbf{S}_{-1}$, S_1 can be written

$$S_1 = a_1 + \widetilde{\theta} + \theta_1^{\perp} + \varepsilon_1,$$

where $\widetilde{\theta}$ is independent of $\theta_1^{\perp}$ and ε_1 and has density function $f_{\widetilde{\theta}}$ defined by $f_{\widetilde{\theta}}(t) \equiv f^Q_{\overline{\theta}}(\overline{\theta} = t \mid \mathbf{S}_{-1})$. We first show that $f_{\widetilde{\theta}}$ is a C^1, strictly positive, strictly log-concave function with bounded derivative. By Bayes' rule,

$$f_{\widetilde{\theta}}(t) = \frac{f_{\overline{\theta}}(t)\prod_{i>1} g^Q_i(S_i \mid \overline{\theta} = t)}{g^Q(\mathbf{S}_{-1})} = \frac{f_{\overline{\theta}}(t)\prod_{i>1} f_{\varepsilon+\theta^{\perp}}(S_i - t - a_i)}{g^Q(\mathbf{S}_{-1})},$$

where $f_{\varepsilon+\theta^{\perp}}$ is the convolution of $f_{\theta^{\perp}}$ and f_{ε}. Since $f_{\theta^{\perp}}$ and f_{ε} are both C^1, strictly positive, strictly log-concave functions with bounded derivatives, so is $f_{\varepsilon+\theta^{\perp}}$. It follows immediately that $f_{\widetilde{\theta}}$ is a strictly positive, C^1 function with bounded derivative. Further, taking logarithms yields

$$\log f_{\widetilde{\theta}}(t) = \log f_{\overline{\theta}}(t) - \log g^Q(\mathbf{S}_{-1}) + \sum_{i>1} \log f_{\varepsilon+\theta^{\perp}}(S_i - t - a_i).$$

Hence $\log f_{\widetilde{\theta}}$ is a sum of constant and strictly concave functions, meaning it is strictly concave. Thus $f_{\widetilde{\theta}}$ is strictly log-concave. This means that conditional on $\mathbf{S}_{-1}$, S_1 is the sum of a constant plus the independent random variables ε_1 and $\widetilde{\theta} + \theta_1^{\perp}$, each of which has a C^1, strictly positive, strictly log-concave density function with bounded derivative. So by Lemma B.1, $g^Q_1(S_1 \mid \varepsilon_1, \mathbf{S}_{-1})$ satisfies SMLRP with respect to ε_1. Further, $f^Q_{\theta_1}(\varepsilon_1 \mid \mathbf{S}_{-1}) = f_{\varepsilon}(\varepsilon_1)$ is continuous in ε_1 by assumption. Lemma B.2 then ensures that $F^Q_{\varepsilon_1}(\varepsilon_1 \mid \mathbf{S})$ is a C^1 function of (ε_1, S_1) and $\frac{\partial}{\partial S_1} F^Q_{\varepsilon_1}(\varepsilon_1 \mid \mathbf{S}) < 0$ everywhere.

By definition, $\mathbb{E}[\theta_1 \mid \mathbf{S}]$ is equal to

$$\mathbb{E}[\theta_1 \mid \mathbf{S}] = \int_{-\infty}^{\infty} \theta_1 \, dF^Q_{\theta_1}(\theta_1 \mid \mathbf{S}).$$

We will perform a change of measure to expect over quantiles of θ_1 rather than θ_1 itself. Fix $\mathbf{S}_{-1}$. The previous paragraphs ensure that $F^Q_{\theta_1}(t \mid \mathbf{S}) - q$ is a C^1 function of (t, S_1, q) everywhere, while Lemma O.3 ensures that the Jacobian of this function wrt to t is $f^Q_{\theta_1}(t \mid \mathbf{S}) > 0$. Then by the implicit function theorem there exists a continuously differentiable quantile function $\phi(q, S_1)$ such that $F^Q_{\theta_1}(\phi(q, S_1) \mid \mathbf{S}) = q$ and

$$\frac{\partial \phi}{\partial S_1}(q, S_1) = -\left[\frac{1}{f^Q_{\theta_1}(t \mid \mathbf{S})} \frac{\partial}{\partial S_1} F^Q_{\theta_1}(t \mid \mathbf{S})\right]_{t=\phi(q,S_1)} > 0$$

for every $q \in (0, 1)$ and $S_1 \in \mathbb{R}$. Changing measure, $\mathbb{E}[\theta_1 \mid \mathbf{S}]$ may be expressed as an expectation over quantiles of θ_1, yielding

$$\mathbb{E}[\theta_1 \mid \mathbf{S}] = \int_0^1 \phi(q, S_1) \, dq.$$

Then for any $\Delta > 0$,

$$\frac{1}{\Delta}\mathbb{E}[\theta_1 \mid S_1 + \Delta, \mathbf{S}_{-1}] - \mathbb{E}[\theta_1 \mid \mathbf{S}] = \int_0^1 \frac{1}{\Delta}(\phi(q, S_1 + \Delta) - \phi(q, S_1))\, dq.$$

By Assumption 3, $\mathbb{E}[\theta_1 \mid \mathbf{S}]$ is differentiable wrt S_1 everywhere. So the limit of each side is well-defined as $\Delta \downarrow 0$. Further, as $\phi(q, S_1)$ is strictly increasing in S_1 for each q, the interior of the integrand is everywhere positive. Then by Fatou's lemma

$$\frac{\partial}{\partial S_1}\mathbb{E}[\theta_1 \mid \mathbf{S}] \geq \int_0^1 \frac{\partial \phi}{\partial S_1}(q, S_1)\, dq > 0.$$

Now, recall that

$$S_1 = a_1 + \theta_1 + \varepsilon_1,$$

so that

$$S_1 = \mathbb{E}[S_1 \mid \mathbf{S}] = a_1 + \mathbb{E}[\theta_1 \mid \mathbf{S}] + \mathbb{E}[\varepsilon_1 \mid \mathbf{S}].$$

Then in particular $\mathbb{E}[\varepsilon_1 \mid \mathbf{S}]$ must be differentiable wrt S_1 given that the remaining terms in the identity are. Very similar work to the previous paragraph then implies that

$$\frac{\partial}{\partial S_1}\mathbb{E}[\varepsilon_1 \mid \mathbf{S}] > 0.$$

Finally, differentiate the identity relating $\mathbb{E}[\theta_1 \mid \mathbf{S}]$ and $\mathbb{E}[\varepsilon_1 \mid \mathbf{S}]$ to obtain

$$1 = \frac{\partial}{\partial S_1}\mathbb{E}[\theta_1 \mid \mathbf{S}] + \frac{\partial}{\partial S_1}\mathbb{E}[\varepsilon_1 \mid \mathbf{S}].$$

Since each term on the right-hand side is strictly positive, each much also be strictly less than 1. □

We next establish that the posterior expectation of the common component of the outcome in each model is increasing in each agent's outcome, with the rate of increase bounded strictly above 0.

Lemma B.7. *For each agent $i \in \{1, ..., N\}$ and outcome-action profile* $(\mathbf{S}, \mathbf{a})$:

- *In the quality linkage model,* $\frac{\partial}{\partial S_i}\mathbb{E}[\overline{\theta} \mid \mathbf{S}] > 0$,
- *In the circumstance linkage model,* $\frac{\partial}{\partial S_i}\mathbb{E}[\overline{\varepsilon} \mid \mathbf{S}] > 0$.

Proof. For convenience, we suppress the dependence of distributions on $\mathbf{a}$ in this proof. We establish the result for the quality linkage model, with the proof for the circumstance linkage model following by nearly identical work. By definition of $\mathbb{E}[\overline{\theta} \mid \mathbf{S}]$,

$$\mathbb{E}[\overline{\theta} \mid \mathbf{S}] = \int \overline{\theta}\, dF^Q_{\overline{\theta}}(\overline{\theta} \mid \mathbf{S}).$$

Now, Lemma B.3 established that $F^Q_{\overline{\theta}}(\overline{\theta} \mid \mathbf{S})$ is a C^1 function of $(\overline{\theta}, S_i)$, and $\frac{\partial}{\partial S_i} F^Q_{\overline{\theta}}(\overline{\theta} \mid \mathbf{S}) < 0$ everywhere. Then the function $F^G_{\overline{\theta}}(\overline{\theta} \mid \mathbf{S}) - q$ is a C^1 function of $(q, \overline{\theta}, S_i)$ with Jacobian $f^Q_{\overline{\theta}}(\overline{\theta} \mid \mathbf{S})$ wrt Q. By Bayes' rule

$$f^Q_{\overline{\theta}}(\overline{\theta} \mid \mathbf{S}) = \frac{f_{\overline{\theta}}(\overline{\theta}) \prod_{i=1}^N g_i(S_i \mid \overline{\theta})}{\int d\overline{\theta}' f_{\overline{\theta}}(\overline{\theta}') \prod_{i=1}^N g_i(S_i \mid \overline{\theta}')},$$

and as $f_{\overline{\theta}}(\overline{\theta})$ and $g_i(S_i \mid \overline{\theta}) = f_{\theta^\perp + \varepsilon}(S_i - \overline{\theta} - a_i)$ are all strictly positive by assumption, $f^Q_{\overline{\theta}}(\overline{\theta} \mid \mathbf{S}) > 0$ everywhere. So fix $\mathbf{S}_{-i}$. Then by the implicit function theorem there exists a C^1 quantile function $\phi(q, S_i)$ such that $F^Q_{\overline{\theta}}(\phi(q, S_i) \mid \mathbf{S}) = q$ everywhere, and

$$\frac{\partial \phi}{\partial S_i}(q, S_i) = -\left[\frac{1}{f^Q_{\overline{\theta}}(\overline{\theta} \mid \mathbf{S})} \frac{\partial}{\partial S_i} F^Q_{\overline{\theta}}(\overline{\theta} \mid \mathbf{S})\right]_{\overline{\theta} = \phi(q, S_i)} > 0.$$

By a change of measure, $\mathbb{E}[\overline{\theta} \mid \mathbf{S}]$ may be expressed as an integral with respect to quantiles of $\overline{\theta}$ as

$$\mathbb{E}[\overline{\theta} \mid \mathbf{S}] = \int_0^1 \phi(q, S_i)\, dq.$$

Then for every $\Delta > 0$,

$$\begin{aligned} &\frac{1}{\Delta}\left(\mathbb{E}[\overline{\theta} \mid \mathbf{S}_{-i}, S_i = s_i + \Delta] - \mathbb{E}[\overline{\theta} \mid \mathbf{S}_{-i}, S_i = s_i]\right) \\ &= \int_0^1 \frac{1}{\Delta}\left(\phi(q, s_i + \Delta) - \phi(q, s_i)\right) dq. \end{aligned}$$

Assumption 3 guarantees that $\frac{\partial}{\partial S_i}\mathbb{E}[\overline{\theta} \mid \mathbf{S}]$ exists. So the limit of each side as $\Delta \downarrow 0$ is well-defined. Further, since $\phi(q, S_i)$ is strictly increasing, the integrand on the rhs is well-defined. Then by Fatou's lemma,

$$\frac{\partial}{\partial S_i}\mathbb{E}[\overline{\theta} \mid \mathbf{S}] \geq \int_0^1 \frac{\partial \phi}{\partial S_i}(q, S_i)\, dq > 0.$$

□

We next establish how the posterior expectation of each agent's type changes as some other agent's outcome shifts. Note that the result depends critically on the model. For simplicity we consider how agent 1's variables shift as agent N's outcome changes. By symmetry an analogous result holds for any pair of agents.

Lemma B.8. *For every outcome-action profile* $(\mathbf{S}, \mathbf{a})$,

$$\frac{\partial}{\partial S_N}\mathbb{E}[\theta_1 \mid \mathbf{S}; \mathbf{a}] > 0$$

in the quality linkage model, while

$$\frac{\partial}{\partial S_N}\mathbb{E}[\theta_1 \mid \mathbf{S};\mathbf{a}] < 0$$

in the circumstance linkage model.

Proof. For convenience, we suppress the dependence of distributions on $\mathbf{a}$ in this proof. By definition $\mathbb{E}[\theta_1 \mid \mathbf{S}]$ is given by

$$\mathbb{E}[\theta_1 \mid \mathbf{S}] = \int_{-\infty}^{\infty} \theta_1 \, dF^M_{\theta_1}(\theta_1 \mid \mathbf{S}).$$

Fix $\mathbf{S}_{-N}$. By Lemma O.4, $F^M_{\theta_1}(\theta_1 \mid \mathbf{S})$ is a C^1 function of (S_N, θ_1), and so $F^M_{\theta_1}(\theta_1 \mid \mathbf{S}) - q$ is a C^1 function of (S_N, θ_1, q) with Jacobian $f^M_{\theta_1}(\theta_1 \mid \mathbf{S})$ wrt θ_1. By Lemma O.3 the Jacobian is strictly positive everywhere, hence by the implicit function theorem there exists a C^1 quantile function $\phi(q, S_N)$ satisfying $F^M_{\theta_1}(\phi(q, S_1) \mid \mathbf{S}) = q$ everywhere, with derivative

$$\frac{\partial \phi}{\partial S_N}(q, S_N) = -\left[\frac{1}{f^M_{\theta_1}(\theta_1 \mid \mathbf{S})}\frac{\partial}{\partial S_N}F^M_{\theta_1}(\theta_1 \mid \mathbf{S})\right]_{\theta_1 = \phi(q, S_N)}.$$

By Lemma B.5, $\frac{\partial}{\partial S_N}F^Q_{\theta_1}(\theta_1 \mid \mathbf{S}) < 0$ everywhere while $\frac{\partial}{\partial S_N}F^C_{\theta_1}(\theta_1 \mid \mathbf{S}) > 0$ everywhere. Hence $\frac{\partial \phi}{\partial S_N}(q, S_N) > 0$ everywhere in the quality linkage model, while $\frac{\partial \phi}{\partial S_N}(q, S_N) < 0$ everywhere in the circumstance linkage model.

By a change of variables, $\mathbb{E}[\theta_1 \mid \mathbf{S}]$ may be expressed as an integral over quantiles of θ_1 as

$$\mathbb{E}[\theta_1 \mid \mathbf{S}] = \int_0^1 \phi(q, S_N)\, dq.$$

Consider first the quality linkage model. For every $\Delta > 0$ we have

$$\begin{aligned}&\frac{1}{\Delta}\left(\mathbb{E}[\theta_1 \mid \mathbf{S}_{-N}, S_N = s_N + \Delta] - \mathbb{E}[\theta_1 \mid \mathbf{S}_{-N}, S_N = s_N]\right)\\ =& \int_0^1 \frac{1}{\Delta}(\phi(q, s_N + \Delta) - \phi(s_N))\, dq,\end{aligned}$$

where the integrand is strictly positive for every $\Delta > 0$ given that $\frac{\partial \phi}{\partial S_N}(q, S_N) > 0$ everywhere. By Assumption 3, $\mathbb{E}[\theta_1 \mid \mathbf{S}]$ is differentiable wrt S_N everywhere, so the limits of both sides must exist as $\Delta \downarrow 0$. Then by Fatou's lemma,

$$\frac{\partial}{\partial S_N}\mathbb{E}[\theta_1 \mid \mathbf{S}] \geq \int_0^1 \frac{\partial \phi}{\partial S_N}(q, S_N)\, dq > 0.$$

Analogously, in the circumstance linkage model

$$\begin{aligned}&-\frac{1}{\Delta}\left(\mathbb{E}[\theta_1 \mid \mathbf{S}_{-N}, S_N = s_N + \Delta] - \mathbb{E}[\theta_1 \mid \mathbf{S}_{-N}, S_N = s_N]\right)\\ =& \int_0^1 -\frac{1}{\Delta}(\phi(q, s_N + \Delta) - \phi(s_N))\, dq,\end{aligned}$$

where the integrand is again positive and the limits of both sides exist. Then by Fatou's lemma

$$-\frac{\partial}{\partial S_N}\mathbb{E}[\theta_1 \mid \mathbf{S}] \geq -\int_0^1 \frac{\partial \phi}{\partial S_N}(q, S_N)\, dq > 0,$$

or equivalently

$$\frac{\partial}{\partial S_N}\mathbb{E}[\theta_1 \mid \mathbf{S}] < 0.$$

□

C Proofs for Section 3 (Exogenous Entry)

C.1 Equilibrium Characterization

In this section we establish that there exists a unique equilibrium to the exogenous-entry model, which is characterized by the first-order condition described in the body of the paper. Fix a population size N, and assume all agents in the segment enter in the first period. For every $\alpha \in \mathbb{R}_+^N$ and $\Delta \geq -\alpha_1$, define

$$\mu(\Delta;\alpha) \equiv \mathbb{E}[\mathbb{E}[\theta_1 \mid \mathbf{S}; \mathbf{a} = \alpha] \mid \mathbf{a} = (\alpha_1 + \Delta, \alpha_{-1})]$$

to be agent 1's expected second-period payoff from exerting effort $\alpha_1 + \Delta$ when the principal expects each agent $i \in \{1, ..., N\}$ to exert effort α_i.

Lemma C.1. *The value function $\mu(\Delta;\alpha)$ and its derivatives satisfy the following properties:*

(a) $\mu(\Delta;\alpha)$ is independent of α and is continuous and strictly increasing in Δ.

(b) $\mu'(\Delta;\alpha)$ exists, is continuous in Δ, and satisfies $0 < \mu'(\Delta;\alpha) < 1$ for every Δ.

(c) $D^+\mu'(\Delta;\alpha) \leq K$ for every Δ.[26]

Proof. Fix a model $M \in \{Q, C\}$. The quantity $\mu(\Delta;\alpha)$ can be written explicitly as

$$\mu(\Delta;\alpha) = \int dG^M(\mathbf{S} = \mathbf{s} \mid \mathbf{a} = (\alpha_1 + \Delta, \alpha_{-1}))\, \mathbb{E}[\theta \mid \mathbf{S} = \mathbf{s}; \mathbf{a} = \alpha].$$

Further,

$$\mathbb{E}[\theta \mid \mathbf{S} = \mathbf{s}; \mathbf{a} = \alpha] = \int \theta_1\, dF^M_{\theta_1}(\theta_1 \mid \mathbf{S} = \mathbf{s}; \mathbf{a} = \alpha),$$

[26]Given a function $f : \mathbb{R} \to \mathbb{R}$, the Dini derivative D^+ is a generalization of the derivative existing for arbitrary functions and defined by $D^+f(x) = \limsup_{h\downarrow 0}(f(x+h) - f(x))/h$. When f is differentiable at a point x, $D^+f(x) = f'(x)$.

and by Bayes' rule

$$f^M_{\theta_1}(\theta_1 \mid \mathbf{S} = \mathbf{s}; \mathbf{a} = \alpha) = \frac{g^M(\mathbf{S} = \mathbf{s} \mid \theta_1; \mathbf{a} = \alpha) f_\theta(\theta)}{g^M(\mathbf{S} = \mathbf{s} \mid \mathbf{a} = \alpha)}.$$

Since effort affects the outcome as an additive shift, $g^M(\mathbf{S} = \mathbf{s} \mid \mathbf{a} = \alpha) = g^M(\mathbf{S} = \mathbf{s} - \alpha \mid \mathbf{a} = \mathbf{0})$ and $g^M(\mathbf{S} = \mathbf{s} \mid \theta_1; \mathbf{a} = \alpha) = g^M(\mathbf{S} = \mathbf{s} - \alpha \mid \theta_1; \mathbf{a} = \mathbf{0})$. So

$$\begin{aligned} f^M_{\theta_1}(\theta \mid \mathbf{S} = \mathbf{s}; \mathbf{a} = \alpha) &= \frac{g^M(\mathbf{S} = \mathbf{s} - \alpha \mid \theta; \mathbf{a} = \mathbf{0}) f_\theta(\theta)}{g^M(\mathbf{S} = \mathbf{s} - \alpha \mid \mathbf{a} = \mathbf{0})} \\ &= f^M_{\theta_1}(\theta_1 \mid \mathbf{S} = \mathbf{s} - \alpha; \alpha = \mathbf{0}). \end{aligned}$$

Thus

$$\mathbb{E}[\theta_1 \mid \mathbf{S} = \mathbf{s}; \mathbf{a} = \alpha] = \int \theta_1 \, dF^M_{\theta_1}(\theta_1 \mid \mathbf{S} = \mathbf{s} - \alpha; \mathbf{a} = \mathbf{0}) = \mathbb{E}[\theta_1 \mid \mathbf{S} = \mathbf{s} - \alpha; \mathbf{a} = \mathbf{0}].$$

Then $\mu(\Delta; \alpha)$ may be equivalently written

$$\mu(\Delta; \alpha) = \int dG^M(\mathbf{S} = \mathbf{s} - \alpha \mid \mathbf{a} = (\Delta, \mathbf{0})) \, \mathbb{E}[\theta_1 \mid \mathbf{S} = \mathbf{s} - \alpha; \mathbf{a} = \mathbf{0}].$$

Using the change of variables $\mathbf{s}' = \mathbf{s} - \alpha$ then reveals that $\mu(\Delta; \alpha) = \mu(\Delta; \mathbf{0})$, so μ is indeed independent of α.

Now fix Δ and $\Delta' < \Delta$. Since effort affects the outcome as an additive shift, $G^M(\mathbf{S} = \mathbf{s} \mid \mathbf{a} = (\alpha_1 + \Delta, \alpha_{-1})) = G^M(\mathbf{S} = (s_1 - (\Delta - \Delta'), \mathbf{s}_{-1}) \mid \mathbf{a} = (\alpha_1 + \Delta, \alpha_{-1}))$ for every s_1. Then defining a change of variables via $s'_1 = s_1 - (\Delta - \Delta')$ and $\mathbf{s}'_{-i} = \mathbf{s}_{-i}$, the previous integral expression for $\mu(\Delta; \alpha)$ may be equivalently written

$$\mu(\Delta; \alpha) = \int dG^M(\mathbf{S} = \mathbf{s}' \mid \mathbf{a} = (\alpha_1 + \Delta', \alpha_{-1})) \, \mathbb{E}[\theta_1 \mid \mathbf{S} = (s'_1 + (\Delta - \Delta'), \mathbf{s}'_{-1}); \mathbf{a} = \alpha].$$

Now, by Assumption 3 $\frac{\partial}{\partial S_1}\mathbb{E}[\theta_1 \mid \mathbf{S}; \mathbf{a}]$ exists and is continuous everywhere, and Lemma B.6 established that $0 < \frac{\partial}{\partial S_1}\mathbb{E}[\theta_1 \mid \mathbf{S}; \mathbf{a}] < 1$ everywhere. Hence by the Leibniz integral rule $\mu'(\Delta; \alpha)$ exists and

$$\mu'(\Delta; \alpha) = \int dG^M(\mathbf{S} = \mathbf{s}' \mid \mathbf{a} = \alpha) \, \frac{\partial}{\partial \Delta}\mathbb{E}[\theta_1 \mid \mathbf{S} = (s'_1 + \Delta, \mathbf{s}'_{-1}); \mathbf{a} = \alpha],$$

and in particular $0 < \mu'(\Delta; \alpha) < 1$. An immediate corollary is that $\mu(\Delta; \alpha)$ is continuous and strictly increasing everywhere. Further, the dominated convergence theorem implies that $\mu'(\Delta; \alpha)$ is continuous in Δ everywhere.

Next, by Assumption 6

$$\frac{\partial^2}{\partial \Delta^2}\mathbb{E}[\theta_1 \mid \mathbf{S} = (s'_1 + \Delta, \mathbf{s}'_{-1}); \mathbf{a} = \alpha]$$

exists and is bounded in the interval $(-\infty, K]$ everywhere. Then for each $\delta > 0$ and $(\mathbf{s}, \mathbf{a}, \Delta)$, the mean value theorem implies that

$$\frac{1}{\delta}\left(\frac{\partial}{\partial\Delta}\mathbb{E}[\theta_1 \mid \mathbf{S} = (s_1' + \Delta + \delta, \mathbf{s}_{-1}'); \mathbf{a} = \alpha] - \frac{\partial}{\partial\Delta}\mathbb{E}[\theta_1 \mid \mathbf{S} = (s_1' + \Delta, \mathbf{s}_{-1}'); \mathbf{a} = \alpha]\right)$$
$$= \frac{\partial^2}{\partial\Delta^2}\mathbb{E}[\theta_1 \mid \mathbf{S} = (s_1' + \Delta + \delta', \mathbf{s}_{-1}'); \mathbf{a} = \alpha] \leq K$$

for some $\delta' \in [0, \delta]$. Reverse Fatou's lemma then implies that $D^+\mu'(\Delta; \alpha) \leq K$. □

Lemma C.2. $\mu(\Delta; \alpha) - C(\alpha_1 + \Delta)$ *is a strictly concave function of* Δ *for any* α.

Proof. Fix an α, and define $\phi(\Delta) \equiv \mu(\Delta; \alpha) - C(\alpha_1 + \Delta)$. By Lemma C.1, ϕ' exists and is continuous everywhere. We establish the necessary and sufficient condition for strict concavity that ϕ' is strictly decreasing. We invoke the basic monotonicity theorem from analysis that any function f which is continuous and satisfies $D^+f \geq 0$ everywhere is nondecreasing everywhere. We apply this result to $-\mu'(\Delta; \alpha) + K\Delta$. Using basic properties of the Dini derivatives D^+ and D_+, we have $D^+(-\mu'(\Delta; \alpha)) = -D_+\mu'(\Delta; \alpha) \geq -D^+\mu'(\Delta; \alpha)$. Then since $K\Delta$ is differentiable and $D^+\mu'(\Delta; \alpha) \leq K$ from Lemma C.1, we have $D^+(-\mu'(\Delta; \alpha) + K\Delta) = D^+(-\mu'(\Delta; \alpha)) + K \geq 0$. So $\mu'(\Delta; \alpha) - K\Delta$ is nonincreasing everywhere. So choose any Δ and $\Delta' > \Delta$. Then

$$\phi'(\Delta') = \mu'(\Delta'; \alpha) - K\Delta' + K\Delta' - C'(\alpha_1 + \Delta') \leq \mu'(\Delta; \alpha) + K(\Delta' - \Delta) - C'(\alpha_1 + \Delta').$$

But also by Assumption 6, $C''(\alpha_1 + \Delta'') > K$ for every $\Delta'' \in (\Delta, \Delta')$, so $C'(\alpha_1 + \Delta') > C'(\alpha_1 + \Delta) + K(\Delta' - \Delta)$. Thus

$$\phi'(\Delta') < \mu'(\Delta; \alpha) - C'(\alpha_1 + \Delta) = \phi'(\Delta),$$

as desired. □

Proposition C.1. *There exists a unique equilibrium action profile characterized by* $a_i = a_i^*(N)$ *for each player* i*, where* $a_i^*(N)$ *is the unique solution to*

$$\mu'(0; \mathbf{a}^*(N)) = C'(a^*(N)).$$

Proof. Lemma C.1 established that $\mu'(0; \mathbf{a}^*(N))$ is well-defined, independent of $a^*(N)$, and bounded in the interval $[0, 1]$. Then as C' is continuous, strictly increasing, and satisfies $C'(0) = 0$ and $C'(\infty) = \infty$, there exists a unique solution to the stated first-order condition. This solution constitutes an equilibrium so long as $\Delta = 0$ maximizes the objective function $\mu(\Delta; \mathbf{a}^*(N)) - C(a^*(N) + \Delta)$, which is guaranteed by the fact, established in Lemma C.2, that this function is strictly concave in Δ. □

Define $\mu_N(\Delta) \equiv \mu(\Delta; \mathbf{a}^*(N))$ and $MV(N) \equiv \mu'_N(0)$ for each N. When we wish to make the model clear, we will write $MV_M(N)$ for $M \in \{Q, C\}$. An immediate implication of Lemma C.1 is that $0 < MV(N) < 1$ for all N. We conclude this appendix by establishing that these bounds also hold strictly in the limit as $N \to \infty$.

Lemma C.3. $0 < \lim_{N\to\infty} MV(N) < 1$.

Proof. Consider first the quality linkage model. The proof of Lemma 1 establishes that $\lim_{N\to\infty} MV_Q(N) = MV_Q(\infty)$, where $MV_Q(\infty)$ is the equilibrium marginal value of effort in a one-agent model where the common component $\overline{\theta}$ is observed by the principal. In this case the agent's equilibrium expected value of distortion is

$$\mu_\infty(\Delta; a^*(\infty)) = \mathbb{E}[\overline{\theta}] + \mathbb{E}[\mathbb{E}[\theta_1^\perp \mid \widetilde{S}_1; a_1 = a^*(\infty)] \mid a_1 = a^*(\infty) + \Delta],$$

where $\widetilde{S}_1 \equiv a_1 + \theta_1^\perp + \varepsilon_1$. Since the contribution of $\overline{\theta}$ to the agent's payoff is not influenced by effort, it has no incentive effect. The marginal value of effort in this setting is then just the marginal value of effort in a one-agent model where the agent's type has density $f_{\theta^\perp}$. As this distribution satisfies the same regularity conditions as f_θ, the reasoning establishing that $0 < MV_Q(1) < 1$ immediately implies that $0 < MV_Q(\infty) < 1$ as well.

Now consider the circumstance linkage model. In this model the proof of Lemma 1 establishes that $\lim_{N\to\infty} MV_C(N) = MV_C(\infty)$, where $MV_C(\infty)$ is the equilibrium marginal value of effort in a one-agent model where the common component $\overline{\varepsilon}$ is observed by the principal. In this case the agent's equilibrium expected value of distortion is

$$\mu_\infty(\Delta; a^*(\infty)) = \mathbb{E}[\mathbb{E}[\theta_1 \mid \widetilde{S}_1; a_1 = a^*(\infty)] \mid a_1 = a^*(\infty) + \Delta],$$

where $\widetilde{S}_1 \equiv a_1 + \theta_1 + \varepsilon_1^\perp$. The marginal value of effort in this setting is then just the marginal value of effort in a one-agent model where the noise distribution has density $f_{\varepsilon^\perp}$. As this distribution satisfies the same regularity conditions as f_ε, the reasoning establishing that $0 < MV_C(1) < 1$ immediately implies that $0 < MV_C(\infty) < 1$ as well. □

C.2 Proof of Lemma 1

Throughout this proof, we will without loss of generality consider agent 1's problem. To compare results across segments of differing sizes, we will consider there to be a single underlying vector $\mathbf{S} = (S_1, S_2, ...)$ of outcomes for a countably infinite set of agents, with the N-agent model corresponding to observation of the outcomes of the first N agents. We will write $\mathbf{a}^*(N)$ to indicate the N-vector with entries $a^*(N)$, and similarly $\mathbf{a}^*(N+1)$ to indicate the $N+1$-vector with entries $a^*(N+1)$. Given any finite or countably infinite vector $\mathbf{x}$ with at least j elements, we will use $\mathbf{x}_{i:j}$ to indicate the subvector of $\mathbf{x}$ consisting of elements i through j. For the distribution function of the outcome vector $\mathbf{S}_{i:j}$, we will write $G^M_{i:j}$.

C.2.1 Monotonicity in N

We first establish the monotonicity claims of the lemma. Fix a model $M \in \{Q, C\}$ and a segment size N. By definition, the expected value of distortion $\mu_N(\Delta)$ is

$$\mu_N(\Delta) = \int dG^M_{1:N}(\mathbf{S}_{1:N} = \mathbf{s}_{1:N} \mid \mathbf{a}_{1:N} = (a^*(N) + \Delta, \mathbf{a}^*(N)_{2:N})) \\ \times \mathbb{E}[\theta_1 \mid \mathbf{S}_{1:N} = \mathbf{s}_{1:N}; \mathbf{a}_{1:N} = \mathbf{a}^*(N)].$$

By Lemma C.1, the value of distortion is independent of the action vector expected by the principal, so we may equivalently write

$$\mu_N(\Delta) = \int dG^M_{1:N}(\mathbf{S}_{1:N} = \mathbf{s}_{1:N} \mid \mathbf{a}_{1:N} = (a^*(N+1) + \Delta, \mathbf{a}^*(N+1)_{2:N})) \\ \times \mathbb{E}[\theta_1 \mid \mathbf{S}_{1:N} = \mathbf{s}_{1:N}; \mathbf{a}_{1:N} = \mathbf{a}^*(N+1)_{1:N}] \qquad \text{(C.1)}$$

replacing $a^*(N)$ everywhere with $a^*(N+1)$. Further, the additive structure of the model implies that the distribution function $G^M_{1:N}$ satisfies the identity

$$G^M_{1:N}(\mathbf{S}_{1:N} = \mathbf{s}_{1:N} \mid \mathbf{a}_{1:N}) = G^M_{1:N}(\mathbf{S}_{1:N} = \mathbf{s}_{1:N} + \mathbf{b}_{1:N} \mid \mathbf{a}_{1:N} + \mathbf{b}_{1:N})$$

for any outcome realization $\mathbf{s}_{1:N}$, action vector $\mathbf{a}_{1:N}$, and shift vector $\mathbf{b}_{1:N}$. Then taking $\mathbf{b}_{1:N} = (-\Delta, \mathbf{0}_{1:N-1})$, the representation of $\mu_N(\Delta)$ in (C.1) may be rewritten

$$\mu_N(\Delta) = \int dG^M_{1:N}(\mathbf{S}_{1:N} = \mathbf{s}'_{1:N} \mid \mathbf{a}_{1:N} = \mathbf{a}^*(N+1)_{1:N}) \\ \times \mathbb{E}[\theta_1 \mid \mathbf{S}_{1:N} = (s'_1 + \Delta, \mathbf{s}'_{2:N}); \mathbf{a}_{1:N} = \mathbf{a}^*(N+1)_{1:N}],$$

where we have changed variables to the integrator $\mathbf{s}'_{1:N} = \mathbf{s}_{1:N} + \mathbf{b}_{1:N}$.

Meanwhile, the value of distortion with $N+1$ agents is

$$\mu_{N+1}(\Delta) = \int dG^M_{1:N+1}(\mathbf{S}_{1:N+1} = \mathbf{s}_{1:N+1} \mid \mathbf{a}_{1:N+1} = (a^*(N+1) + \Delta, \mathbf{a}^*(N+1)_{2:N+1})) \\ \times \mathbb{E}[\theta_1 \mid \mathbf{S}_{1:N+1} = \mathbf{s}_{1:N+1}; \mathbf{a}_{1:N+1} = \mathbf{a}^*(N+1)].$$

Using the same transformation as in the N-agent model, this expression may be equivalently written

$$\mu_{N+1}(\Delta) = \int dG^M_{1:N+1}(\mathbf{S}_{1:N+1} = \mathbf{s}'_{1:N+1} \mid \mathbf{a}_{1:N+1} = \mathbf{a}^*(N+1)) \\ \times \mathbb{E}[\theta_1 \mid \mathbf{S}_{1:N+1} = (s'_1 + \Delta, \mathbf{s}'_{2:N+1}); \mathbf{a}_{1:N+1} = \mathbf{a}^*(N+1)].$$

For the remainder of the proof, all distributions will be conditioned on the action profile $a_{1:N+1} = \mathbf{a}^*(N+1)$, so conditioning of distributions on actions will be suppressed.

To compare the expressions for $\mu_N(\Delta)$ and $\mu_{N+1}(\Delta)$ just derived, we use the law of iterated expectations. In the N-agent model we have

$$\mathbb{E}[\theta_1 \mid \mathbf{S}_{1:N} = (s_1 + \Delta, \mathbf{s}_{2:N})] = \int dG^M_{N+1}(S_{N+1} = s_{N+1} \mid \mathbf{S}_{1:N} = (s_1 + \Delta, \mathbf{s}_{2:N})) \\ \times \mathbb{E}[\theta_1 \mid \mathbf{S}_{1:N+1} = (s_1 + \Delta, \mathbf{s}_{2:N+1})].$$

So

$$\mu_N(\Delta) = \int dG^M_{1:N}(\mathbf{S}_{1:N} = \mathbf{s}_{1:N}) \\ \times \int dG^M_{N+1}(S_{N+1} = s_{N+1} \mid \mathbf{S}_{1:N} = (s_1 + \Delta, \mathbf{s}_{2:N})) \\ \times \mathbb{E}[\theta_1 \mid \mathbf{S}_{1:N+1} = (s_1 + \Delta, \mathbf{s}_{2:N+1})].$$

Meanwhile in the $N+1$-agent model the law of iterated expectations may be applied to the unconditional expectation over $\mathbf{S}_{1:N+1}$ to obtain

$$\mu_{N+1}(\Delta) = \int dG^M_{1:N+1}(\mathbf{S}_{1:N} = \mathbf{s}_{1:N}) \\ \times \int dG^M_{N+1}(S_{N+1} = s_{N+1} \mid \mathbf{S}_{1:N} = \mathbf{s}_{1:N}) \\ \times \mathbb{E}[\theta_1 \mid \mathbf{S}_{1:N+1} = (s_1 + \Delta, \mathbf{s}_{2:N+1})].$$

So define a function ψ by

$$\psi(\delta_1, \delta_2, \mathbf{s}_{1:N}) \equiv \int dG^M_{N+1}(S_{N+1} = s_{N+1} \mid \mathbf{S}_{1:N} = (s_1 + \delta_1, \mathbf{s}_{2:N})) \\ \times \mathbb{E}[\theta_1 \mid \mathbf{S}_{1:N+1} = (s_1 + \delta_2, \mathbf{s}_{2:N+1})].$$

Then the values of distortion with N and $N+1$ agents may be written in the common form

$$\mu_N(\Delta) = \int dG^M_{1:N}(\mathbf{S}_{1:N})\, \psi(\Delta, \Delta, \mathbf{S}_{1:N})$$

while

$$\mu_{N+1}(\Delta) = \int dG^M_{1:N}(\mathbf{S}_{1:N})\, \psi(0, \Delta, \mathbf{S}_{1:N}).$$

Then for any $\Delta > 0$,

$$\frac{1}{\Delta}(\mu_N(\Delta) - \mu_{N+1}(\Delta)) = \int dG^M_{1:N}(\mathbf{S}_{1:N})\, \frac{1}{\Delta}(\psi(\Delta, \Delta, \mathbf{S}_{1:N}) - \psi(0, \Delta, \mathbf{S}_{1:N})).$$

Now, as $MV(N) = \mu'_N(0)$ and $MV(N+1) = \mu'_{N+1}(0)$ both exist and are finite by Lemma C.1, it follows that

$$MV(N) - MV(N+1) = \lim_{\Delta \downarrow 0} \frac{1}{\Delta}(\mu_N(\Delta) - \mu) - \lim_{\Delta \downarrow 0} \frac{1}{\Delta}(\mu_{N+1}(\Delta) - \mu) \\ = \lim_{\Delta \downarrow 0} \frac{1}{\Delta}(\mu_N(\Delta) - \mu_{N+1}(\Delta))$$

exists, so that

$$MV(N) - MV(N+1) = \lim_{\Delta \downarrow 0} \int dG^M_{1:N}(\mathbf{S}_{1:N}) \frac{1}{\Delta}(\psi(\Delta, \Delta, \mathbf{S}_{1:N}) - \psi(0, \Delta, \mathbf{S}_{1:N})),$$

and in particular the limit on the rhs also exists. To bound the right-hand side and complete the proof, we analyze the behavior of $\frac{1}{\Delta}(\psi(\Delta, \Delta, \mathbf{S}_{1:N}) - \psi(0, \Delta, \mathbf{S}_{1:N}))$ as Δ tends to zero.

Consider first the quality linkage model. Using the law of total probability, we may re-write $\psi(\delta_1, \delta_2, \mathbf{s}_{1:N})$ as

$$\begin{aligned}\psi(\delta_1, \delta_2, \mathbf{s}_{1:N}) = &\int dF^Q_{\overline{\theta}}(\overline{\theta} \mid \mathbf{S}_{1:N} = (s_1 + \delta_1, \mathbf{s}_{2:N})) \\ &\times \int dG^Q_{N+1}(S_{N+1} = s_{N+1} \mid \overline{\theta}, \mathbf{S}_{1:N} = (s_1 + \delta, \mathbf{s}_{2:N})) \\ &\times \mathbb{E}[\theta_1 \mid \mathbf{S}_{1:N+1} = (s_1 + \delta_2, \mathbf{s}_{2:N+1})].\end{aligned}$$

As S_{N+1} is independent of $\mathbf{S}_{1:N}$ conditional on $\overline{\theta}$, this is equivalently

$$\begin{aligned}\psi(\delta_1, \delta_2, \mathbf{s}_{1:N}) = &\int dF^Q_{\overline{\theta}}(\overline{\theta} \mid \mathbf{S}_{1:N} = (s_1 + \delta_1, \mathbf{s}_{2:N})) \\ &\times \int dG^Q_{N+1}(S_{N+1} = s_{N+1} \mid \overline{\theta}) \, \mathbb{E}[\theta_1 \mid \mathbf{S}_{1:N+1} = (s_1 + \delta_2, \mathbf{s}_{2:N+1})].\end{aligned}$$

Inverting

$$q = G^Q_{N+1}(S_{N+1} = s_{N+1} \mid \overline{\theta}) = F_{\theta^\perp + \varepsilon}(s_{N+1} - \overline{\theta} - a^*(N+1))$$

yields the quantile function $s_{N+1} = F^{-1}_{\theta^\perp + \varepsilon}(q) + \overline{\theta} + a^*(N+1)$, so by a change of variables ψ may be equivalently written

$$\begin{aligned}\psi(\delta_1, \delta_2, \mathbf{s}_{1:N}) = &\int dF^Q_{\overline{\theta}}(\overline{\theta} \mid \mathbf{S}_{1:N} = (s_1 + \delta_1, \mathbf{s}_{2:N})) \\ &\times \int_0^1 dq \, \mathbb{E}[\theta_1 \mid \mathbf{S}_{1:N+1} = (s_1 + \delta_2, \mathbf{s}_{2:N}, F^{-1}_{\theta^\perp + \varepsilon}(q) + \overline{\theta} + a^*(N+1))].\end{aligned}$$

Now fix $\mathbf{s}_{1:N}$, and write the integrand of this representation as

$$\zeta(\overline{\theta}, \delta, q) \equiv \mathbb{E}[\theta_1 \mid \mathbf{S}_{1:N+1} = (s_1 + \delta, \mathbf{s}_{2:N}, F^{-1}_{\theta^\perp + \varepsilon}(q) + \overline{\theta} + a^*(N+1))].$$

By Lemma B.5, $F^Q_{\overline{\theta}}(\overline{\theta} \mid \mathbf{S}_{1:N} = (s_1 + \delta, \mathbf{s}_{2:N}))$ is a C^1 function of $(\overline{\theta}, \delta)$ satisfying $\frac{\partial}{\partial \delta} F^Q_{\overline{\theta}}(\overline{\theta} \mid \mathbf{S}_{1:N} = (s_1 + \delta, \mathbf{s}_{2:N})) < 0$ everywhere. Then $F^Q_{\overline{\theta}}(\overline{\theta} \mid \mathbf{S}_{1:N} = (s_1 + \delta, \mathbf{s}_{2:N})) - q'$ is a C^1 function of $(q', \overline{\delta}, \delta)$, with Jacobian $f^Q_{\overline{\theta}}(\overline{\theta} \mid \mathbf{S}_{1:N} = (s_1 + \delta, \mathbf{s}_{2:N}))$ wrt $\overline{\theta}$. By Lemma O.3 this Jacobian is strictly positive everywhere. Then by the implicit function theorem there exists a C^1 quantile function $\phi(q', \delta)$ satisfying $F^Q_{\overline{\theta}}(\phi(q', \delta) \mid \mathbf{S}_{1:N} = (s_1 + \delta, \mathbf{s}_{2:N})) = q'$ for all (q', δ_1) and

$$\frac{\partial \phi}{\partial \delta}(q', \delta) = -\left[\frac{1}{f^Q_{\overline{\theta}}(\overline{\theta} \mid \mathbf{S}_{1:N} = (s_1 + \delta, \mathbf{s}_{2:N}))} \frac{\partial}{\partial \delta} F^Q_{\overline{\theta}}(\overline{\theta} \mid \mathbf{S}_{1:N} = (s_1 + \delta, \mathbf{s}_{2:N}))\right]_{\overline{\theta} = \phi(q', \delta)} > 0.$$

Then by a further change of variables, $\psi(\delta_1, \delta_2, \mathbf{s}_{1:N})$ may be written

$$\psi(\delta_1, \delta_2, \mathbf{s}_{1:N}) = \int_0^1 dq' \int_0^1 dq\, \zeta(\phi(q', \delta_1), \delta_2, q).$$

By Lemma B.8 $\frac{\partial}{\partial S_{N+1}} \mathbb{E}[\theta_1 \mid \mathbf{S}_{1:N+1}] > 0$ everywhere. Since further $\partial\phi/\partial\delta > 0$, it follows that

$$\zeta(\phi(q', \Delta), \Delta, q) > \zeta(\phi(q', 0), \Delta, q)$$

for all (q, q') and every $\Delta > 0$. Hence $\frac{1}{\Delta}(\psi(\Delta, \Delta, \mathbf{s}_{1:N}) - \psi(0, \Delta, \mathbf{s}_{1:N})) > 0$ for every $\Delta > 0$. This argument holds independent of the choice of $\mathbf{s}_{1:N}$. Thus Fatou's lemma implies

$$MV(N) - MV(N+1) \geq \int dG^M_{1:N}(\mathbf{S}_{1:N}) \liminf_{\Delta \downarrow 0} \frac{1}{\Delta}(\psi(\Delta, \Delta, \mathbf{S}_{1:N}) - \psi(0, \Delta, \mathbf{S}_{1:N})).$$

A further application of Fatou's lemma yields

$$\begin{aligned} &\liminf_{\Delta \downarrow 0} \frac{1}{\Delta}(\psi(\Delta, \Delta, \mathbf{S}_{1:N}) - \psi(0, \Delta, \mathbf{S}_{1:N})) \\ \geq &\int_0^1 dq' \int_0^1 dq \lim_{\Delta \downarrow 0} \frac{1}{\Delta}(\zeta(\phi(q', \Delta), \Delta, q) - \zeta(\phi(q', 0), \Delta, q)). \end{aligned}$$

Recall that by Assumption 3, $\frac{\partial}{\partial S_i} \mathbb{E}[\theta_1 \mid \mathbf{S}_{1:N+1}]$ exists and is continuously differentiable in $\mathbf{S}_{1:N+1}$ for every i. Thus $\mathbb{E}[\theta_1 \mid \mathbf{S}_{1:N+1}]$ is totally differentiable wrt $\mathbf{S}_{1:N+1}$ everywhere. So write the integrand of the previous expression for $\liminf_{\Delta \downarrow 0} \frac{1}{\Delta}(\psi(\Delta, \Delta, \mathbf{S}_{1:N}) - \psi(0, \Delta, \mathbf{S}_{1:N}))$ as

$$\begin{aligned} &\frac{1}{\Delta}(\zeta(\phi(q', \Delta), \Delta, q) - \zeta(\phi(q', 0), \Delta, q)) \\ = &\frac{1}{\Delta}(\zeta(\phi(q', \Delta), \Delta, q) - \zeta(\phi(q', 0), 0, q)) - \frac{1}{\Delta}(\zeta(\phi(q', 0), \Delta, q, q') - \zeta(\phi(q', 0), 0, q)). \end{aligned}$$

Taking $\Delta \downarrow 0$ and using the chain rule yields

$$\begin{aligned} &\lim_{\Delta \downarrow 0} \frac{1}{\Delta}(\zeta(\phi(q', \Delta), \Delta, q) - \zeta(\phi(q', 0), \Delta, q)) \\ = &\frac{\partial \zeta}{\partial \overline{\theta}}(\phi(q', 0), 0, q)\frac{\partial \phi}{\partial \delta}(q', 0) + \frac{\partial \zeta}{\partial \delta}(\phi(q', 0), 0, q) - \frac{\partial \zeta}{\partial \delta}(\phi(q', 0), 0, q) \\ = &\frac{\partial \zeta}{\partial \overline{\theta}}(\phi(q', 0), 0, q)\frac{\partial \phi}{\partial \delta}(q', 0). \end{aligned}$$

The fact that $\frac{\partial}{\partial S_{N+1}}[\theta_1 \mid \mathbf{S}_{1:N+1}] > 0$ implies that $\frac{\partial \zeta}{\partial \overline{\theta}}(\phi(q', 0), 0, q) > 0$, and as previously noted $\partial\phi/\partial\delta > 0$. It follows that this limit is strictly positive. Thus

$$\liminf_{\Delta \downarrow 0} \frac{1}{\Delta}(\psi(\Delta, \Delta, \mathbf{S}_{1:N}) - \psi(0, \Delta, \mathbf{S}_{1:N})) > 0$$

everywhere, meaning in turn $MV(N) > MV(N+1)$.

The result for the circumstance linkage model follows from very similar work. The one difference in the analysis is that in the circumstance linkage model Lemma B.8 implies that $\frac{\partial}{\partial S_{N+1}}\mathbb{E}[\theta_1 \mid \mathbf{S}_{1:N+1}] < 0$ everywhere, so that in this model

$$\lim_{\Delta \downarrow 0} \frac{1}{\Delta}(\zeta(\phi(q', 0), \Delta, q) - \zeta(\phi(q', \Delta), \Delta, q)) > 0$$

and

$$\frac{1}{\Delta}(\psi(0, \Delta, \mathbf{s}_{1:N}) - \psi(\Delta, \Delta, \mathbf{s}_{1:N})) > 0$$

everywhere. Then by Fatou's lemma

$$\begin{aligned} MV(N+1) - MV(N) &= \lim_{\Delta \downarrow 0} \int dG^M_{1:N}(\mathbf{S}_{1:N}) \frac{1}{\Delta}(\psi(0, \Delta, \mathbf{S}_{1:N} - \psi(\Delta, \Delta, \mathbf{S}_{1:N})) \\ &\geq \int dG^M_{1:N}(\mathbf{S}_{1:N}) \liminf_{\Delta \downarrow 0} \frac{1}{\Delta}(\psi(0, \Delta, \mathbf{S}_{1:N} - \psi(\Delta, \Delta, \mathbf{S}_{1:N})) > 0, \end{aligned}$$

or $MV(N) < MV(N+1)$.

C.2.2 The $N \to \infty$ limit

Consider a limiting model in which the principal observes a countably infinite vector of outcomes $\mathbf{S} = (S_1, S_2, ...)$. By the law of large numbers, in the quality linkage model this means that the principal perfectly infers $\overline{\theta}$, while in the circumstance linkage model the principal perfectly infers $\overline{\varepsilon}$. Define $\mu(\Delta; \alpha)$ analogously to the finite-population case. In each model,reasoning very similar to the proof of Lemma C.1 implies that $\mu'(0, \alpha)$ exists, is independent of α, and lies in $[0, 1]$. So there exists a unique, finite $a^*(\infty)$ satisfying $\mu'(0; \mathbf{a}^*(\infty)) = C'(a^*(\infty))$. Define $\mu_\infty(\Delta) \equiv \mu(\Delta; \mathbf{a}^*(\infty))$ and $MV(\infty) \equiv \mu'_\infty(0)$ in each model. Lemma C.3 establishes that $0 < MV(\infty) < 1$. We will show that $\lim_{N\to\infty} MV(N) = MV(\infty)$. Lemma C.3 establishes that this result implies $0 < \lim_{N\to\infty} MV(N) < 1$.

To prove the result, we will need the ability to change measure between the distribution of outcomes at the equilibrium action profile, and one in which a single agent, without loss agent 1, deviates to a different action. For each model, define a reference probability space $(\Omega, \mathcal{F}, \mathcal{P}^{\mathbf{a}})$, containing all relevant random variables for arbitrary segment sizes. For the quality linkage model this space supports the latent types $\overline{\theta}, \theta_1^\perp, \theta_2^\perp, ...$ and shocks $\varepsilon_1, \varepsilon_2, ...$ as well as the outcomes $S_1, S_2, ...$ Similarly, in the circumstance linkage model the space supports the latent types $\theta_1, \theta_2, ...$, shocks $\overline{\varepsilon}, \varepsilon_1^\perp, \varepsilon_2^\perp, ...$, and outcomes $S_1, S_2, ...$ In each model the probability measure $\mathcal{P}^{\mathbf{a}}$ depends on the vector of agent actions $\mathbf{a} = (a_1, a_2, ...)$, as the distributions of the outcomes depend on the actions.

We will use $\mathcal{F}^\infty$ to denote the σ-algebra generated by the full vector of outcomes $S_1, S_2, ...$ Note that by the LLN all latent types may be taken to be measurable with respect to $\mathcal{F}^\infty$.

Finally, for each segment size N, we will let $\mathcal{P}^{*N}$ denote the restriction of the measure $\mathcal{P}^{\mathbf{a}^*(N)}$ to $(\Omega, \mathcal{F}^\infty)$, and similarly let $\mathcal{P}^{\Delta,N}$ denote the restriction of the measure $\mathcal{P}^{(a^*(N)+\Delta,\mathbf{a}^*(N))}$ to $(\Omega, \mathcal{F}^\infty)$. These measures represent the distributions over outcomes induced when all agents take actions $a^*(N)$ and when agent 1 deviates to action $a^*(N) + \Delta$, respectively.

Lemma C.4. *The Radon-Nikodym derivative for the change of measure from* $(\Omega, \mathcal{F}^\infty, \mathcal{P}^{*N})$ *to* $(\Omega, \mathcal{F}^\infty, \mathcal{P}^{\Delta,N})$ *is*

$$\frac{d\mathcal{P}^{\Delta,N}}{d\mathcal{P}^{*N}} = \frac{g_1^Q(S_1 \mid \overline{\theta}; a_1 = a^*(N) + \Delta)}{g_1^Q(S_1 \mid \overline{\theta}; a_1 = a^*(N))}$$

in the quality linkage model and

$$\frac{d\mathcal{P}^{\Delta,N}}{d\mathcal{P}^{*N}} = \frac{g_1^C(S_1 \mid \overline{\varepsilon}; a_1 = a^*(N) + \Delta)}{g_1^C(S_1 \mid \overline{\varepsilon}; a_1 = a^*(N))}$$

in the circumstance linkage model.

Proof. For convenience we suppress the dependence of distributions on all actions other than a_1 in this proof. We derive the derivative for the quality linkage model, with the expression for the circumstance linkage model following from nearly identical work. Fix any $\mathcal{F}^\infty$-measurable random variable X. Then there exists a measurable function $x : \mathbb{R}^\infty \to \mathbb{R}$ such that $X = x(\mathbf{S})$ a.s. Thus

$$\begin{aligned}
&\mathbb{E}[X \mid a_1 = a^*(N) + \Delta] \\
= &\int dF_{\overline{\theta}}(\overline{\theta})\, dG_1^Q(S_1 \mid \overline{\theta}; a_1 = a^*(N) + \Delta)\, dG_{-1}^Q(\mathbf{S}_{-1} \mid \overline{\theta}, S_1; a_1 = a^*(N) + \Delta) \\
&\quad \times x(\mathbf{S}).
\end{aligned}$$

As $\mathbf{S}_{-1}$ is independent of S_1 conditional on $\overline{\theta}$ in the quality linkage model, $G_{-1}^Q(\mathbf{S}_{-1} \mid \overline{\theta}, S_1; a_1 = a^*(N) + \Delta) = G_{-1}^Q(\mathbf{S}_{-1} \mid \overline{\theta})$. So this expression may be equivalently written

$$\begin{aligned}
&\mathbb{E}[X \mid a_1 = a^*(N) + \Delta] \\
= &\int dF_{\overline{\theta}}(\overline{\theta})\, dG_1^Q(S_1 \mid \overline{\theta}; a_1 = a^*(N) + \Delta)\, dG_{-1}^Q(\mathbf{S}_{-1} \mid \overline{\theta})\, x(\mathbf{S}) \\
= &\int dF_{\overline{\theta}}(\overline{\theta})\, dG_1^Q(S_1 \mid \overline{\theta}; a_1 = a^*(N))\, dG_{-1}^Q(\mathbf{S}_{-1} \mid \overline{\theta}) \\
&\quad \times \frac{g_1^Q(S_1 \mid \overline{\theta}; a_1 = a^*(N) + \Delta)}{g_1^Q(S_1 \mid \overline{\theta}; a_1 = a^*(N))} x(\mathbf{S}) \\
= &\, \mathbb{E}\left[\frac{g_1^Q(S_1 \mid \overline{\theta}; a_1 = a^*(N) + \Delta)}{g_1^Q(S_1 \mid \overline{\theta}; a_1 = a^*(N))} X \mid a_1 = a^*(N) \right].
\end{aligned}$$

As this argument holds for arbitrary $\mathcal{F}^\infty$-measurable X, it must be that

$$\frac{d\mathcal{P}^{\Delta,N}}{d\mathcal{P}^{*N}} = \frac{g_1^Q(S_1 \mid \overline{\theta}; a_1 = a^*(N) + \Delta)}{g_1^Q(S_1 \mid \overline{\theta}; a_1 = a^*(N))}.$$

□

To establish the desired limiting result, we will prove that for any Δ and N,

$$|\mu_N(\Delta) - \mu_\infty(\Delta)| \le \kappa_N(\Delta)\frac{\beta}{\sqrt{N}},$$

where

$$\kappa_N(\Delta) \equiv \left(\mathbb{E}\left[\left(\frac{d\mathcal{P}^{\Delta,N}}{d\mathcal{P}^{*N}} - 1\right)^2 \,\middle|\, \mathbf{a} = \mathbf{a}^*(N)\right]\right)^{1/2}$$

and β is a finite constant independent of N and Δ whose value depends on the model. The following lemma establishes several important properties of κ_N.

Lemma C.5. *$\kappa_N(\Delta)$ is independent of N, $\kappa_N(0) = 0$, $\overline{\kappa}'_{N,+}(0) = \limsup_{\Delta\downarrow 0} \kappa_N(\Delta)/\Delta < \infty$.*

Proof. We prove the theorem for the quality linkage model, with nearly identical work establishing the result for the circumstance linkage model. Note that when $\Delta = 0$, $d\mathcal{P}^{\Delta,N}/d\mathcal{P}^{*N} = 1$, and so trivially $\kappa_N(0) = 0$. To see that $\kappa_N(\Delta)$ is independent of N, note that the distribution of each outcome satisfies the translation invariance property $G_i^Q(S_i = s_i \mid \overline{\theta}; a_i = \alpha) = G_i^Q(S_i = s_i - \alpha \mid \overline{\theta}; a_i = 0)$ for any s_i and α. So $\kappa_N(\Delta)$ may be written

$$\begin{aligned}\kappa_N(\Delta) &= \int dF_{\overline{\theta}}(\overline{\theta})\, dG_1^Q(S_1 = s_1 \mid \overline{\theta}; a_1 = a^*(N)) \left(\frac{g_1^Q(S_1 = s_1 \mid \overline{\theta}; a_1 = a^*(N) + \Delta)}{g_1^Q(S_1 = s_1 \mid \overline{\theta}; a_1 = a^*(N))} - 1\right)^2 \\ &= \int dF_{\overline{\theta}}(\overline{\theta})\, dG_1^Q(S_1 = s_1 - a^*(N) \mid \overline{\theta}; a_1 = 0) \left(\frac{g_1^Q(S_1 = s_1 - a^*(N) \mid \overline{\theta}; a_1 = \Delta)}{g_1^Q(S_1 = s_1 - a^*(N) \mid \overline{\theta}; a_1 = 0)} - 1\right)^2\end{aligned}$$

So perform a change of variables to $s_1' \equiv s_1 - a^*(N)$ to obtain the representation

$$\kappa_N(\Delta) = \int dF_{\overline{\theta}}(\overline{\theta})\, dG_1^Q(S_1 = s_1' \mid \overline{\theta}; a_1 = 0) \left(\frac{g_1^Q(S_1 = s_1' \mid \overline{\theta}; a_1 = \Delta)}{g_1^Q(S_1 = s_1' \mid \overline{\theta}; a_1 = 0)} - 1\right)^2,$$

which is independent of N, as desired.

Now, let $\xi \equiv \theta_1^\perp + \varepsilon_1$. Let f_ξ be the convolution of $f_{\theta^\perp}$ and f_ε. Then for any Δ, $g_1^Q(S_1 \mid$

$\overline{\theta}; a_1 = a^*(N) + \Delta) = f_\xi(S_1 - \overline{\theta} - a^*(N) - \Delta) = f_\xi(\xi - \Delta)$ under the measure $\mathcal{P}^{*N}$. Hence

$$\begin{aligned}\kappa_N(\Delta) = &\left(\mathbb{E}\left[\left(\frac{d\mathcal{P}^{\Delta,N}}{d\mathcal{P}^{*N}} - 1\right)^2 \,\middle|\, \mathbf{a} = \mathbf{a}^*(N)\right]\right)^{1/2} \\ = &\left(\mathbb{E}\left[\left(\frac{f_\xi(\xi - \Delta)}{f_\xi(\xi)} - 1\right)^2 \,\middle|\, \mathbf{a} = \mathbf{a}^*(N)\right]\right)^{1/2} \\ = &\int dF_\xi(\xi) \left(\frac{f_\xi(\xi - \Delta) - f_\xi(\xi)}{f_\xi(\xi)}\right)^2\end{aligned}$$

We must therefore show that the limit

$$\begin{aligned}\limsup_{\Delta\downarrow 0} \frac{1}{\Delta}\kappa(\Delta) = &\limsup_{\Delta\downarrow 0} \frac{1}{\Delta}\left(\int dF_\xi(\xi) \left(\frac{f_\xi(\xi - \Delta) - f_\xi(\xi)}{f_\xi(\xi)}\right)^2\right)^{1/2} \\ = &\left(\limsup_{\Delta\downarrow 0} \int dF_\xi(\xi) \frac{1}{\Delta^2}\left(\frac{f_\xi(\xi - \Delta) - f_\varepsilon(\xi)}{f_\varepsilon(\xi)}\right)^2\right)^{1/2}\end{aligned}$$

exists and is finite. By Assumption 4, for Δ sufficiently close to 0 there exists a non-negative, integrable function $J(\cdot)$ such that

$$\frac{1}{\Delta^2}\left(\frac{f_\xi(\xi - \Delta) - f_\xi(\xi)}{f_\xi(\xi)}\right)^2 \leq J(\xi)$$

for all ξ. Then by reverse Fatou's lemma,

$$\begin{aligned}\limsup_{\Delta\downarrow 0} \int dF_\xi(\xi) \frac{1}{\Delta^2}\left(\frac{f_\xi(\xi - \Delta) - f_\xi(\xi)}{f_\xi(\xi)}\right)^2 &\leq \int dF_\xi(\xi) \limsup_{\Delta\downarrow 0} \frac{1}{\Delta^2}\left(\frac{f_\xi(\xi - \Delta) - f_\xi(\xi)}{f_\xi(\xi)}\right)^2 \\ &\leq \int dF_\xi(\xi)\, J(\xi) < \infty,\end{aligned}$$

as desired. □

The bound on $|\mu_N(\Delta) - \mu_\infty(\Delta)|$ just claimed impies the desired result because for $\Delta > 0$ it may be rewritten

$$|(\mu_N(\Delta) - \mu)/\Delta - (\mu_\infty(\Delta) - \mu)/\Delta| \leq \frac{\kappa_N(\Delta) - \kappa_N(0)}{\Delta} \frac{\beta}{\sqrt{N}},$$

and thus by taking $\Delta \downarrow 0$ the inequality

$$|\mu'_N(0) - \mu'_\infty(0)| \leq \overline{\kappa}'_{N,+}(0)\frac{\beta}{N}$$

must hold. Then as $\overline{\kappa}'_{N,+}(0)$ is finite and independent of N, $\mu'_N(0) \to \mu'_\infty(0)$ as $N \to \infty$, as desired.

We now derive the claimed bound. To streamline notation, we will write $\mathbb{E}^{*N}$ to represent expectations conditioning on $\mathbf{a} = \mathbf{a}^*(N)$, and $\mathbb{E}^{\Delta,N}$ to represent expectations conditioning on $a_1 = a^*(N) + \Delta$ and $\mathbf{a}_{2:N} = \mathbf{a}^*(N)_{1:N-1}$. Note first that the expected value of the principal's posterior estimate of θ_1 is a function only of the size of agent 1's distortion Δ, but *not* of the equilibrium action inference. Thus

$$\begin{aligned}\mu_\infty(\Delta) &= \mathbb{E}[\mathbb{E}[\theta_1 \mid \mathbf{S}; \mathbf{a} = \mathbf{a}^*(\infty)] \mid \mathbf{a} = (a^*(\infty) + \Delta, \mathbf{a}^*(\infty))] \\ &= \mathbb{E}[\mathbb{E}[\theta_1 \mid \mathbf{S}; \mathbf{a} = \mathbf{a}^*(N)] \mid \mathbf{a} = (a^*(N) + \Delta, \mathbf{a}^*(N))] = \mathbb{E}^{\Delta,N}[\mathbb{E}^{*N}[\theta_1 \mid \mathbf{S}]].\end{aligned}$$

So we may write

$$\mu_N(\Delta) - \mu_\infty(\Delta) = \mathbb{E}^{\Delta,N}[\mathbb{E}^{*N}[\theta_1 \mid \mathbf{S}_{1:N}] - \mathbb{E}^{*N}[\theta_1 \mid \mathbf{S}]].$$

Now, performing a change of measure,

$$\begin{aligned}&\mathbb{E}^{\Delta,N}[\mathbb{E}^{*N}[\theta_1 \mid \mathbf{S}_{1:N}] - \mathbb{E}^{*N}[\theta_1 \mid \mathbf{S}]] \\ &= \mathbb{E}^{*N}\left[\frac{d\mathcal{P}^{\Delta,N}}{d\mathcal{P}^{*N}}\left(\mathbb{E}^{*N}[\theta_1 \mid \mathbf{S}_{1:N}] - \mathbb{E}^{*N}[\theta_1 \mid \mathbf{S}]\right)\right] \\ &= \mathbb{E}^{*N}\left[\left(\frac{d\mathcal{P}^{\Delta,N}}{d\mathcal{P}^{*N}} - 1\right)\left(\mathbb{E}^{*N}[\theta_1 \mid \mathbf{S}_{1:N}] - \mathbb{E}^{*N}[\theta_1 \mid \mathbf{S}]\right)\right] \\ &\quad + \mathbb{E}^{*N}[\mathbb{E}^{*N}[\theta_1 \mid \mathbf{S}_{1:N}] - \mathbb{E}^{*N}[\theta_1 \mid \mathbf{S}]] \\ &= \mathbb{E}^{*N}\left[\left(\frac{d\mathcal{P}^{\Delta,N}}{d\mathcal{P}^{*N}} - 1\right)\left(\mathbb{E}^{*N}[\theta \mid \mathbf{S}_{1:N}] - \mathbb{E}^{*N}[\theta_1 \mid \mathbf{S}]\right)\right],\end{aligned}$$

with the last line following by the law of iterated expectations. Then by an application of the Cauchy-Schwarz inequality,

$$|\mu_N(\Delta) - \mu_\infty(\Delta)| \leq \kappa_N(\Delta)\left(\mathbb{E}^{*N}\left[\left(\mathbb{E}^{*N}[\theta_1 \mid \mathbf{S}_{1:N}] - \mathbb{E}^{*N}[\theta_1 \mid \mathbf{S}]\right)^2\right]\right)^{1/2}.$$

We will bound the right-hand side for the quality linkage model, with the result for the circumstance linkage model following by nearly identical work.

Define the family of random variables $\widehat{\theta}_N(z) \equiv \mathbb{E}^{*N}[\theta_1 \mid S_1, \overline{\theta} = z]$ for $z \in \mathbb{R}$. Note that $\widehat{\theta}_1(\overline{\theta}) = \mathbb{E}^{*N}[\theta_1 \mid \mathbf{S}]$, as $\mathbf{S}$ allows the principal to perfectly infer $\overline{\theta}$, and θ_1 is independent of the vector of outcomes $\mathbf{S}_{-1}$ conditional on $\overline{\theta}$. Further, $\mathbb{E}^{*N}[\theta_1 \mid \mathbf{S}_{1:N}] = \mathbb{E}^{*N}[\mathbb{E}^{*N}[\theta_1 \mid \mathbf{S}] \mid \mathbf{S}_{1:N}]$ is the mean-square minimizing estimator of $\widehat{\theta}_N(\overline{\theta})$ conditional on the performance vector $\mathbf{S}_{1:N}$. Another estimator of $\widehat{\theta}_N(\overline{\theta})$ is $\widehat{\theta}_N\left(\widetilde{\theta}_N\right)$, where

$$\widetilde{\theta}_N \equiv \frac{1}{N}\sum_{i=1}^{N}(S_i - \mu^\perp),$$

with $\mu^\perp = \mathbb{E}[\theta_i^\perp]$. So

$$\mathbb{E}^{*N}\left[\left(\mathbb{E}^{*N}[\theta_1 \mid \mathbf{S}_{1:N}] - \mathbb{E}^{*N}[\theta_1 \mid \mathbf{S}]\right)^2\right] \leq \mathbb{E}^{*N}\left[\left(\widehat{\theta}_N\left(\widetilde{\theta}_N\right) - \mathbb{E}^{*N}[\theta_1 \mid \mathbf{S}]\right)^2\right].$$

Given that shifts in $\overline{\theta}$ affect the outcome S_i additively, $\mathbb{E}^{*N}[\theta_1 \mid S_1 = s_1, \overline{\theta} = z] = \mathbb{E}^{*N}[\theta_1 \mid S_1 = s_1 - z, \overline{\theta} = 0]$ for every s_1 and z. The proof of Lemma C.3 establishes that $\mathbb{E}^{*N}[\theta_1 \mid S_1, \overline{\theta}]$ is differentiable with respect to S_1 and uniformly bounded in $(0,1)$ everywhere. Hence $\widehat{\theta}_N(z)$ is differentiable and $\widehat{\theta}'_N(z) \in (-1, 0)$ for all z. Thus by the fundamental theorem of calculus,

$$|\widehat{\theta}_N(\widetilde{\theta}_N) - \widehat{\theta}_N(\overline{\theta})| = \left|\int_{\overline{\theta}}^{\widetilde{\theta}_N} \widehat{\theta}'_N(z)\, dt\right| \leq \int_{\overline{\theta}}^{\widetilde{\theta}_N} |\widehat{\theta}'_N(z)|\, dz \leq |\widetilde{\theta}_N - \overline{\theta}|.$$

Further note that

$$\widetilde{\theta}_N - \overline{\theta} = \frac{1}{N}\sum_{i=1}^{N}(\theta_i^\perp - \mu^\perp + \varepsilon_i),$$

which has mean 0 and variance $(\sigma^2_{\theta^\perp} + \sigma^2_\varepsilon)/N$ given that $\theta_i^\perp$ and ε_i are independent. So

$$\mathbb{E}^{*N}\left[\left(\mathbb{E}^{*N}[\theta_1 \mid \mathbf{S}_{1:N}] - \mathbb{E}^{*N}[\theta_1 \mid \mathbf{S}]\right)^2\right] \leq \frac{\sigma^2_{\theta^\perp} + \sigma^2_\varepsilon}{N},$$

implying the desired bound with $\beta = \sqrt{\sigma^2_{\theta^\perp} + \sigma^2_\varepsilon}$.

D Proofs for Section 4 (Main Results)

D.1 Proofs of Theorems 1 and 2

Opt-In Equilibrium. In any pure-strategy equilibrium in which all agents opt-in, the equilibrium effort level a^* must satisfy two conditions:

$$MV(N) = C'(a^*) \tag{D.1}$$

$$R + \mu - C(a^*) \geq 0 \tag{D.2}$$

The expression in (D.1) guarantees that an agent who opts-in cannot strictly gain by deviating to a different effort choice. This is identical to the condition used in the exogenous entry model to solve for equilibrium. The expression in (D.2) guarantees that agents cannot profitably deviate to opting-out.

The marginal value $MV(N)$ is independent of a^*, and C' is strictly monotone. Thus (D.1) pins down a unique effort level $a^* = C'^{-1}(MV(N))$. Since C is everywhere increasing, the conditions in (D.1) and (D.2) can be simultaneously satisfied if and only if $0 \leq C'^{-1}[MV(N)] \leq a^{**} \equiv C^{-1}(R + \mu)$, or equivalently,

$$0 = C'(0) \leq MV(N) \leq C'(a^{**})$$

noting that C'^{-1} is everywhere increasing.

By Assumption 7, $R + \mu > C(a^*(1))$. Since the cost function C has positive first and second derivatives, $R + \mu > C(a^*(1))$ and $R + \mu = C(a^{**})$ imply that $a^*(1) < a^{**}$, which further implies $C'(a^*(1)) < C'(a^{**})$. By Lemma 1, $MV(1) = MV_Q(1) \geq MV_Q(N)$. Thus

$$MV_Q(N) \leq MV_Q(1) = C'(a^*(1)) \leq C'(a^{**}),$$

and a symmetric all opt-in equilibrium exists in the quality linkage model. In contrast, in the circumstance linkage model,

$$MV_C(N) \geq MV_C(1) = C'(a^*(1)) \tag{D.3}$$

so the inequality $MV_C(N) \leq C'(a^{**})$ is not guaranteed to hold. An opt-in equilibrium exists if and only if N is sufficiently small; specifically, $N \leq N^*$ where

$$N^* \equiv \sup\{N \ : \ MV_C(N) \leq C'(a^{**})\}.$$

(It is possible that N^* is infinite if $MV_C(N) \leq C'(a^{**})$ for all N.)

Finally, for the parameters $N \leq N^*$ where an opt-in equilibrium exists in both models, it is possible to rank equilibrium effort levels as follows: Define a^*_C and a^*_Q to be the respective equilibrium effort levels. Then, since $MV_C(N) \geq MV(1) \geq MV_Q(N)$ for all N,

$$a^*_C = C'^{-1}(MV_C(N)) \geq C^{-1}(MV_Q(N)) = a^*_Q$$

so equilibrium effort is higher in the circumstance linkage model.

Opt-Out Equilibrium. Under the imposed refinement on the principal's off-equilibrium belief about the agent's action, the optimal action conditional on entry is $a^*(1)$. Thus in an all opt-out equilibrium, the equilibrium action a^* must satisfy

$$R + \mu - C(a^*(1)) < 0 \tag{D.4}$$

which violates Assumption 7. There are no pure-strategy equilibria in either model in which all agents choose to opt-out.

Mixed Equilibrium. For any probability $p \in [0, 1]$ and $M \in \{T, C\}$, let

$$MV_M(p, N) = \mathbb{E}\left[\left(MV_M(\widetilde{N} + 1\right) \mid \widetilde{N} \sim \text{Binomial}(N - 1, p)\right]$$

be the expected marginal impact for agent i of exerting additional effort beyond the principal's expectation, when agent i opts-in and all other agents opt-in with independent probability p. Note that because $MV_C(N)$ is increasing in N, and increasing p shifts up the distribution of $\widetilde{N}$ in the FOSD sense, $MV_C(p,N)$ is increasing in p. Further, because increasing p shifts $\Pr(\widetilde{N} \leq n)$ strictly downward for every $n < N-1$, this monotonicity is strict whenever $MV_C(n)$ is not constant over the range $\{1,..,N\}$. For the same reasons, $MV_C(p,N)$ is increasing in N for fixed p, and strictly increasing whenever $p \in (0,1)$ and $MV_C(n)$ is not constant over $\{1,...,N\}$.

In a mixed equilibrium, the equilibrium effort level a^* and probability p assigned to opting-in must jointly satisfy

$$R + \mu - C(a^*) = 0. \tag{D.5}$$

$$MV(p,N) = C'(a^*). \tag{D.6}$$

The expression in (D.5) pins down the equilibrium action, which is identical to the action defined as a^{**} above. Moreover, $C'(a)$ is independent of both the mixing probability p and also the fixed segment size N. Therefore an equilibrium exists if and only if $MV(p,N) = C'(a^{**})$ for some $p \in [0,1]$. But for all $p \in [0,1]$,

$$MV_Q(p,N) \leq \max_{1 \leq N' \leq N} MV_Q(N') = MV_Q(1) = C'(a^*(1)) < C'(a^{**})$$

using that MV_Q is a decreasing function of N (Lemma 1). Thus the quality linkage model does not admit a strictly mixed equilibrium.

Similarly if $MV_C(N) < C'(a^{**})$, then

$$MV_C(p,N) \leq \max_{1 \leq N' \leq N} MV_C(N') = MV_C(N) < C'(a^{**})$$

since MV_C is a strictly increasing function of N (Lemma 1). So there does not exist a strictly mixed equilibrium in the circumstance linkage model either. Indeed, this is exactly the range for N that supports the symmetric all opt-in equilibrium in the circumstance linkage model.

If however $MV(N) \geq C'(a^{**})$, then

$$MV_C(1) = MV_C(0,N) < C'(a^{**}) \leq MV_C(1,N) = MV_C(N).$$

This implies in particular that MV_C is not constant over the range $\{1,...,N\}$, so that $MV_C(p,N)$ is strictly increasing in p. Since $MV_C(p,N)$ is also continuous in p, the intermediate value theorem yields existence of a unique $p^*(N) \in (0,1]$ satisfying $MV_C(p^*(N),N) = C'(a^{**})$.

If $N \leq N^*$, i.e. $MV(N) = C'(a^{**})$, then it must be that $p^*(N) = 1$. Thus in particular the opt-in equilibrium is unique whenever it exists. Otherwise $p^*(N) < 1$, in which case the fact that $MV_C(p,N)$ is strictly increasing in N for fixed $p \in (0,1)$ further implies that $p^*(N)$ must be strictly decreasing in N. Finally, the effort level a^{**} chosen in this equilibrium weakly exceeds the effort level a_C^* chosen in the symmetric opt-in equilibrium in the circumstance linkage model, since $R + \mu \geq C(a_C^*)$ by (D.2), while $R + \mu = C(a^{**})$ by (D.5).

D.2 Proof of Lemma 2

Comparisons between equilibrium actions correspond directly to comparisons of marginal values of effort. It is therefore sufficient to establish that $MV(N) < 1$ for all N, and that $MV_Q(N)$ is decreasing while $MV_C(N)$ is increasing in N, with $\lim_{N\to\infty} MV_C(N) < 1$. These facts in particular imply that $MV_Q(N) \leq MV(1) \leq MV_C(N)$, with $MV(1)$ dictating equilibrium effort in the no-data linkages benchmark. Lemma 1 establishes the desired monotonicity of the marginal value of effort, while the upper bound on MV and the limiting value of MV_C are established in Appendix C.

D.3 Proof of Proposition 2

Suppose all agents in a segment of size N enter and choose action a. Social welfare

$$W(1, a, N) = N \cdot (2\mu + a - C(a))$$

is strictly increasing on $a \in [0, a_{FB})$. Thus the comparison $a_Q^*(N) \leq a_{NDL} < a_{FB}$ immediately implies that for all N, welfare is ranked

$$W_Q(N) \leq W_{NDL}(N)$$

where the inequality is strict for all $N > 1$.

For segment sizes $N < N^*$, the equilibrium action in the circumstance linkage model satisfies $a_C^*(N) \in [a_{NDL}, a_{FB})$ (Theorem 2), so the same argument implies

$$W_{NS}(N) \leq W_C(N)$$

with the inequality strict for $N > 1$. When the segment size $N > N^*$,

$$W_C(N) = N \cdot p(N) \cdot [a^{**} - C(a^{**}) + 2\mu] .$$

Since $p(N) \to 0$ as $N \to \infty$, it follows that for N sufficiently large,

$$W_C(N), W_Q(N) < W_{NDL}(N).$$

E Proofs for Section 5 (Data Sharing, Markets, and Consumer Welfare)

E.1 Proof of Proposition 3

Definition E.1. *The* competitive *transfer for a segment of n consumers is*

$$\overline{R}^*(n) = a^*(n) + \mu \tag{E.1}$$

while the monopolist *transfer is*

$$\underline{R}^*(n) = C[a^*(n)] - \mu \tag{E.2}$$

We first show that in any equilibrium under data sharing, consumers must receive all of the generated surplus.

Lemma E.1. *Consider either the quality linkage or circumstance linkage model. In any equilibrium under data sharing, firms receive zero payoffs, and consumer welfare is*

$$N \times (2\mu + a^*(N) - C(a^*(N)))\,.$$

Proof. Fix any subset of firms F where $|F| \geq 2$. Suppose each firm $f \in F$ sets the competitive transfer $\overline{R}^*(N)$ (as defined in (E.1)), while each firm $f \notin F$ chooses a transfer weakly below $\overline{R}^*(N)$. Consumers opt-out if no firm offers a transfer above $\underline{R}^*(N)$. Otherwise, consumers participate with the firm offering the highest transfer, and exert effort $a^*(N)$.

We now show that this is an equilibrium. By choosing the transfer $\overline{R}^*(N)$, firms $f \in F$ receive a payoff of $-\overline{R}^*(N) + \mu + a^*(N) = 0$ per consumer. They cannot profitably deviate, since reducing their transfer would lose all of their consumers, while increasing their transfer would result in a negative payoff. Firms $f \notin F$ acquire consumers only by setting a transfer strictly above $\overline{R}^*(N)$, which leads to a negative payoff. So there are no profitable deviations for firms. Consumers also have no profitable deviations: participation with any firm in F leads to the same (strictly positive) payoff, while participation with any firm $f \notin F$ involves the same equilibrium effort but a lower transfer. So the described strategies constitute an equilibrium.

Moreover these equilibria are the only equilibria under data sharing. Suppose towards contradiction that some firm f receiving consumers sets $R_f < \overline{R}^*(N)$. If another firm f' offers a transfer $R_{f'} \in (R_f, \overline{R}^*(N)]$, then consumers participating with firm f can profitably deviate to participating with firm f'. If no firms f' offer transfers in the interval $(R_f, \overline{R}^*(N)]$, then firm f can profitably deviate by raising its transfer. So transfers below $\overline{R}^*(N)$ are ruled out for firms receiving consumers. If any firm receiving consumers sets a transfer exceeding $\overline{R}^*(N)$ (which yields a negative payoff), then that firm can strictly profit by deviating to $\overline{R}^*(N)$ (which yields a payoff of zero). So transfers above $\overline{R}^*(N)$ are ruled out as well. In equilibrium, it must therefore be that all firms that receive consumers set transfer $\overline{R}^*(N)$.

Firms not receiving consumers must set transfers weakly below $\overline{R}^*(N)$, or consumers could profitably deviate to one of these firms. Finally, since in equilibrium agents know the number of agents participating with their firm, uniqueness of agent effort follows from arguments given already in the proofs for exogenous transfers (see Section 4). $\square$

We show next that (E.1) is an upper bound on achievable consumer welfare under proprietary data in the circumstance linkage setting. Consider any equilibrium, and let N_f be the number of agents participating with firm f in that equilibrium. We can obtain an upper bound on consumer welfare by evaluating consumer payoffs supposing that all firms set the competitive transfer. Then, each agent interacting with firm f achieves a payoff of $\overline{R}^*(N_f) + \mu - C(a^*_C(N_f))$. But

$$\begin{aligned}\overline{R}^*(N_f) + \mu - C(a^*_C(N_f)) &= a^*_C(N_f) + 2\mu - C[a^*_C(N_f)] \\ &\leq a^*_C(N) + 2\mu - C(a^*_C(N)),\end{aligned}$$

since the function $\xi(n) = a^*_C(n) - C(a^*_C(n))$ is increasing, and $N \geq N_f$. Thus consumer welfare is bounded above by $N \times (a^*_C(N) + 2\mu - C(a^*_C(N)))$ as desired. Since this bound holds uniformly across all allocations of consumers to firms, welfare must be weakly higher under data sharing than in any equilibrium with proprietary data.

Now consider the quality linkage model. We first show that in equilibrium, all consumers must be served by a single firm.

Lemma E.2. *In the quality linkage model under proprietary data, in every equilibrium exactly one firm receives consumers.*

Proof. Suppose towards contradiction that there is an equilibrium in which two firms $f = 1, 2$ set transfers R_f and receive $N_f > 0$ agents. Then, each agent interacting with firm f must choose the effort level $a^*_Q(N_f)$. Agents' IC constraints are described as follows: First,

$$R_1 + \mu - C(a^*_Q(N_1)) \geq R_2 + \mu - C(a^*_Q(N_2 + 1))$$

or agents participating with firm 1 could profitably deviate to participating with firm 2. Likewise it must be that

$$R_2 + \mu - C(a^*_Q(N_2)) \geq R_1 + \mu - C(a^*_Q(N_1 + 1))$$

or agents participating with firm 2 could profitably deviate to participating with firm 1. These displays simplify to

$$\begin{aligned}R_1 - R_2 &\geq C(a^*_Q(N_1)) - C(a^*_Q(N_2 + 1)) \\ R_2 - R_1 &\geq C(a^*_Q(N_2)) - C(a^*_Q(N_1 + 1)).\end{aligned}$$

Summing these inequalities, we have

$$0 \geq C(a^*_Q(N_1)) + C(a^*_Q(N_2)) - C(a^*_Q(N_1 + 1)) - C(a^*_Q(N_2 + 1))$$

But $C(a_Q^*(n))$ is strictly decreasing in n, so the right-hand side of the above display must be strictly positive, leading to a contradiction.

Now suppose no firms receive consumers in equilibrium. If there exists a firm offering a transfer $R > \underline{R}^*(1)$, then it is strictly optimal for a consumer to deviate to interaction with that firm at effort $a^*(1)$. Otherwise, it is strictly optimal for a firm to deviate to any transfer $R \in (\underline{R}_M^*(1), \overline{R}^*(1))$ and receive consumers. □

The lemma says that only one firm receives a strictly positive number of consumers in equilibrium; without loss, let this be firm 1. Consumer welfare is maximized when firm 1 sets the competitive transfer $\overline{R}^*(N)$, in which case consumers receive (E.1), so consumer welfare under proprietary data must be weakly lower than under data sharing, completing our proof.

O For Online Publication

O.1 Distributional Regularity Results

To establish our main results we rely heavily on boundedness and smoothness of various likelihood and posterior distribution functions. In this section we prove a number of technical lemmas ensuring sufficient smoothness of functions invoked in proofs elsewhere.

We first prove a general result showing that log-concave density functions are necessarily bounded.

Lemma O.1. *Let $f : \mathbb{R} \to \mathbb{R}$ be any strictly positive, strictly log-concave function satisfying*[27] $\int_{-\infty}^{\infty} f(x)\,dx < \infty$. *Then f is bounded.*

Proof. As f is bounded below by 0, it suffices to show that it is bounded above. Since $\log f$ is strictly concave everywhere, it is either a strictly monotone function, or else has a global maximizer. Suppose that $\log f$ is strictly increasing everywhere. Then f must be strictly increasing everywhere as well. But then as $f > 0$,

$$\int_{-\infty}^{\infty} f(x)\,dx \geq \int_{0}^{\infty} f(x)\,dx \geq \int_{0}^{\infty} f(0)\,dx = \infty,$$

a contradiction of our assumption. So $\log f$ cannot be strictly increasing everywhere. Suppose instead that $\log f$ is strictly decreasing everywhere. Then f must be strictly decreasing everywhere as well. But then as $f > 0$,

$$\int_{-\infty}^{\infty} f(x)\,dx \geq \int_{-\infty}^{0} f(x)\,dx \geq \int_{0}^{\infty} f(0)\,dx = \infty,$$

another contradiction. So f must have a global maximizer, meaning that it is bounded above as desired. □

Corollary O.1. $f_\theta, f_{\overline{\theta}}, f_{\theta^\perp}, f_\varepsilon, f_{\overline{\varepsilon}}, f_{\varepsilon^\perp}$ *are each bounded.*

The following lemma establishes a set of regularity conditions on a likelihood function sufficient to ensure that its associated posterior distribution function is C^1 in both its arguments. Note that these conditions amount to the regularity conditions imposed in SMLRP, plus a continuity condition on the density of the unobserved variable.

Lemma O.2. *Let X and Y be two random variables for which the density $g(y)$ for Y and the conditional densities $f(x \mid y)$ for $X \mid Y$ exist. Suppose that:*

- *$f(x \mid y)$ is a $C^{1,0}$ function and $g(y)$ is continuous,*

[27]Since $\log f$ is strictly concave everywhere, it is continuous everywhere. Then so is f, meaning that f is a measurable function.

- *$f(x,y)$ and $\frac{\partial}{\partial x} f(x \mid y)$ are both uniformly bounded for all (x,y).*

Then $H(x,y) \equiv \Pr(Y \leq y \mid X = x)$ is a C^1 function of (x,y).

Proof. Let G be the distribution function for y. By Bayes' rule,

$$H(x,y) = \frac{\int_{-\infty}^{y} f(x \mid y')\, dG(y')}{\int_{-\infty}^{\infty} f(x \mid y'')\, dG(y'')}.$$

We first establish continuity of this function. It is sufficient to establish continuity of the numerator and denominator separately. As for the denominator, $f(x \mid y'')$ is continuous in x and uniformly bounded for all (x, y''), so by the dominated convergence theorem the denominator is continuous in x, thus also in (x,y) given its independence of y. As for the numerator, write

$$\int_{-\infty}^{y} f(x \mid y')\, dG(y') = \int_{-\infty}^{\infty} \mathbf{1}\{y' \leq y\} f(x \mid y')\, dG(y').$$

Consider any sequence converging to (x_0, y_0). Given the continuity of $f(x \mid y)$, the integrand converges pointwise G-a.e. to $\mathbf{1}\{y' \leq y_0\} f(x_0 \mid y')$. (The only point of potential nonconvergence is at $y' = y_0$, but since Y is a continuous distribution this point is assigned measure zero under G.) As the integrand is also uniformly bounded above for all (x, y, y'), the dominated convergence theorem ensures that the numerator is continuous in (x,y).

Next, note that $\partial H / \partial y$ exists and is given by

$$\frac{\partial H}{\partial y}(x,y) = \frac{f(x \mid y) g(y)}{\int_{-\infty}^{\infty} f(x \mid y'')\, dG(y'')},$$

which is continuous everywhere given that the denominator is continuous by the argument of the previous paragraph while $f(x \mid y)$ and $g(y)$ are continuous by assumption.

Finally, consider $\partial H / \partial x$. Let $\widehat{H}(x,y) \equiv H(x,y)^{-1} - 1$. Then $\frac{\partial H}{\partial x}(x,y)$ exists and satisfies $\frac{\partial H}{\partial x}(x,y) < 0$ iff $\frac{\partial \widehat{H}}{\partial x}(x,y)$ exists and satisfies $\frac{\partial \widehat{H}}{\partial x}(x,y) > 0$. Note that $\widehat{H}(x,y)$ may be written

$$\widehat{H}(x,y) = \frac{\int_{y}^{\infty} f(x \mid y')\, dG(y')}{\int_{-\infty}^{y} f(x \mid y'')\, dG(y'')}.$$

Because $\frac{\partial}{\partial x} f(x \mid y)$ exists and is uniformly bounded for all x and y, the Leibniz integral rule ensures that this expression is differentiable with respect to x with derivative

$$\frac{\partial \widehat{H}}{\partial x}(x,y) = \frac{\int_{y}^{\infty} \frac{\partial}{\partial x} f(x \mid y')\, dG(y')}{\int_{-\infty}^{y} f(x \mid y'')\, dG(y'')} - \frac{\left(\int_{y}^{\infty} f(x \mid y')\, dG(y')\right)\left(\int_{-\infty}^{y} \frac{\partial}{\partial x} f(x \mid y'')\, dG(y'')\right)}{\left(\int_{-\infty}^{y} f(x \mid y'')\, dG(y'')\right)^2}.$$

With some rearrangement, this may be equivalently written

$$\frac{\partial \widehat{H}}{\partial x}(x,y) = \left(\int_{-\infty}^{y} f(x \mid y'')\, dG(y'')\right)^{-2} \times \int_{y}^{\infty} dG(y') \int_{-\infty}^{y} dG(y'') \left(f(x \mid y'') \frac{\partial}{\partial x} f(x \mid y') - f(x \mid y') \frac{\partial}{\partial x} f(x \mid y'')\right).$$

This function is continuous if both

$$\int_{-\infty}^{\infty} \mathbf{1}\{y'' \leq y\} f(x \mid y'')\, dG(y'')$$

and

$$\int_{-\infty}^{\infty} dG(y') \int_{-\infty}^{\infty} dG(y'')\, \mathbf{1}\{y' \geq y\}\mathbf{1}\{y'' \leq y\} \left(f(x \mid y'') \frac{\partial}{\partial x} f(x \mid y') - f(x \mid y') \frac{\partial}{\partial x} f(x \mid y'') \right)$$

are continuous. We have already seen that the former is continuous, so consider the latter expression. By assumption $f(x \mid y)$ and $\frac{\partial}{\partial x} f(x \mid y)$ are both continuous in (x, y). Thus for any sequence converging to (x_0, y_0), the integrand converges to

$$\mathbf{1}\{y' \geq y_0\}\mathbf{1}\{y'' \leq y_0\} \left(f(x_0 \mid y'') \left. \frac{\partial}{\partial x} f(x \mid y') \right|_{x=x_0} - f(x_0 \mid y') \left. \frac{\partial}{\partial x} f(x \mid y'') \right|_{x=x_0} \right)$$

except possibly at points (y', y'') such that $y' = y_0$ or $y'' = y_0$, a set which is assigned zero measure under $G \times G$ given the continuity of the distribution of Y. Further, since $f(x \mid y)$ and $\frac{\partial}{\partial x} f(x \mid y)$ are both uniformly bounded for all (x, y), so is

$$f(x \mid y) \frac{\partial}{\partial x} f(x \mid y') - f(x \mid y') \frac{\partial}{\partial x} f(x \mid y)$$

for all x, y, y'. Then the dominated convergence theorem ensures that the entire expression converges to its value at (x_0, y_0), as desired. □

The next lemma establishes that the density functions of θ_i and ε_i remain continuous when conditioned on a set of outcomes.

Lemma O.3. *For each model $M \in \{Q, C\}$, agent $i \in \{1, ..., N\}$, and outcome-action profile $(\mathbf{S}, \mathbf{a})$:*

- *The conditional densities $f^M_{\theta_i}(\theta_i \mid \mathbf{S}; \mathbf{a})$ and $f^M_{\theta_i}(\theta_i \mid \mathbf{S}_{-j}; \mathbf{a})$ for each $j \in \{1, ..., N\}$ are strictly positive and continuous in θ_i everywhere,*
- *The conditional densities $f^M_{\varepsilon_i}(\varepsilon_i \mid \mathbf{S}; \mathbf{a})$ and $f^M_{\varepsilon_i}(\varepsilon_i \mid \mathbf{S}_{-j}; \mathbf{a})$ for each $j \in \{1, ..., N\}$ are strictly positive and continuous in ε_i everywhere.*

Proof. Throughout the proof we suppress explicit dependence of distributions on the action profile $\mathbf{a}$. We prove the result for the quality linkage model, with the circumstance linkage model following by permuting the roles of θ_i and ε_i.

Consider first the density of θ_i conditional on $\mathbf{S}$. By Bayes' rule

$$f^Q_{\theta_i}(\theta_i \mid \mathbf{S}) = \frac{g^Q_{1:N}(\mathbf{S} \mid \theta_i) f_\theta(\theta_i)}{g^Q_{1:N}(\mathbf{S})},$$

where

$$g^Q_{1:N}(\mathbf{S} \mid \theta_i) = g^Q_i(S_i \mid \theta_i) \int dF^Q_{\overline{\theta}}(\overline{\theta} \mid \theta_i) \prod_{j \neq i} g^Q_j(S_j \mid \overline{\theta})$$

and

$$g_{1:N}^Q(\mathbf{S}) = \int dF_{\overline{\theta}}(\overline{\theta}) \prod_{j=1}^{N} g_j^Q(S_j \mid \overline{\theta}).$$

As $g_{1:N}^Q(\mathbf{S} \mid \theta_i)$, $g_{1:N}^Q(\mathbf{S})$, and $f_\theta(\theta_i)$ are all strictly positive, so is $f_{\theta_i}^Q(\theta_i \mid \mathbf{S})$. Further, $g_i^Q(S_i \mid \theta_i) = f_\varepsilon(S_i - \theta_i - a_i)$ is continuous in θ_i given the continuity of f_ε. Then $f_{\theta_i}^Q(\theta_i \mid \mathbf{S})$ is continuous in θ_i so long as

$$f_\theta(\theta_i) \int dF_{\overline{\theta}}^Q(\overline{\theta} \mid \theta_i) \prod_{j \neq i} g_j^Q(S_j \mid \overline{\theta}) = \int dF_{\overline{\theta}}(\overline{\theta}) f_{\theta^\perp}(\theta_1 - \overline{\theta}) \prod_{j \neq i} g_j^Q(S_j \mid \overline{\theta})$$

is. As $f_{\theta^\perp}$ is bounded and continuous and $\int dF_{\overline{\theta}}(\overline{\theta}) \prod_{j \neq i} g_j^Q(S_j \mid \overline{\theta}) = g_{1:N}(\mathbf{S})$ is finite, the dominated convergence theorem ensures that this final term is continuous, as desired. The result for the density of θ_i conditional on $\mathbf{S}_{-j}$ for any $j \neq i$ follows from nearly identical work.

Next consider the density of θ_i conditional on $\mathbf{S}_{-i}$. Now Bayes' rule gives

$$f_{\theta_i}^Q(\theta_i \mid \mathbf{S}_{-i}) = \frac{g_{-i}^Q(\mathbf{S}_{-i} \mid \theta_i) f_\theta(\theta_i)}{g_{-i}^Q(\mathbf{S}_{-i})},$$

where

$$g_{-i}^Q(\mathbf{S}_{-i} \mid \theta_i) = \int dF_{\overline{\theta}}^Q(\overline{\theta} \mid \theta_i) \prod_{j \neq i} g_j^Q(S_j \mid \overline{\theta})$$

and

$$g_{-i}^Q(\mathbf{S}_{-i}) = \int dF_{\overline{\theta}}(\overline{\theta}) \prod_{j \neq i} g_j^Q(S_j \mid \overline{\theta}).$$

As each of these terms is strictly positive, so is $f_{\theta_i}^Q(\theta_i \mid \mathbf{S}_{-i})$. Further, $g_{-i}^Q(\mathbf{S}_{-i} \mid \theta_i) f_\theta(\theta_i)$ was already shown to be continuous in the previous paragraph. So $f_{\theta_i}^Q(\theta_i \mid \mathbf{S}_{-i})$ is continuous in θ_i, as desired.

Next, consider the density of ε_i conditional on $\mathbf{S}$. Bayes' rule gives

$$f_{\varepsilon_i}^Q(\varepsilon_i \mid \mathbf{S}) = \frac{g_{1:N}^Q(\mathbf{S} \mid \varepsilon_i) f_\varepsilon(\varepsilon_i)}{g_{1:N}^Q(\mathbf{S})},$$

where

$$g_{1:N}^Q(\mathbf{S} \mid \varepsilon_i) = g_i^Q(S_i \mid \varepsilon_i) \prod_{j \neq i} g_j^Q(S_j).$$

Then as $g_i^Q(S_i \mid \varepsilon_i) = f_\theta(S_i - \varepsilon_i - a_i)$ is continuous in ε_i given the continuity of f_θ, so is $g_{1:N}^Q(\mathbf{S} \mid \varepsilon_i)$. The result for the density of ε_i conditional on $\mathbf{S}_{-j}$ for any $j \neq i$ follows by nearly identical work.

Finally, consider the density of ε_i conditional on $\mathbf{S}_{-i}$. In the quality linkage model ε_i is independent of $\mathbf{S}_{-i}$, so $g_{\varepsilon_i}^Q(\varepsilon_i \mid \mathbf{S}_{-i}) = f_\varepsilon(\varepsilon_i)$, which is strictly positive and continuous by assumption. □

The following pair of lemmas establishes that the posterior distribution functions of the agent's type conditional on the vector of outcomes satisfies a smoothness condition. To economize on notation, the lemma is established with respect to agent 1's latent variables, as the signal of agent N moves. By symmetry an analogous result applies to all other pairs of agents.

Lemma O.4. *For each model $M \in \{Q, C\}$ and outcome-action profile $(\mathbf{S}_{-N}, \mathbf{a})$, $F^M_{\theta_1}(\theta_1 \mid \mathbf{S}; \mathbf{a})$ is a C^1 function of (S_N, θ_1).*

Proof. For convenience, we suppress the dependence of distributions on $\mathbf{a}$ in this proof. Fix $\mathbf{S}_{-N}$. The result follows from Lemma O.2 so long as 1) $f^M_{\theta_1}(\theta_1 \mid \mathbf{S}_{-N})$ is continuous in θ_1, and 2) $g^M_N(S_N \mid \theta_1, \mathbf{S}_{-N})$ is a $C^{1,0}$ function of (S_N, θ_1) and both it and its derivative wrt S_N are uniformly bounded. Lemma O.3 ensures that condition 1 holds, so we need only establish condition 2.

Consider first the quality linkage model. In this case $g^Q_N(S_N \mid \theta_1, \mathbf{S}_{-N}) = g^Q_N(S_N \mid \theta_1, \mathbf{S}_{2:N-1})$, as S_N is independent of S_1 conditional on θ_1. And by the law of total probability,

$$g^Q_N(S_N \mid \theta_1, \mathbf{S}_{2:N-1}) = \int g^Q_N(S_N \mid \overline{\theta}, \theta_1, \mathbf{S}_{2:N-1})\, dF^Q_{\overline{\theta}}(\overline{\theta} \mid \theta_1, \mathbf{S}_{2:N-1}).$$

As S_N is independent of $(\theta_1, \mathbf{S}_{2:N-1})$ conditional on $\overline{\theta}$, this is equivalently

$$g^Q_N(S_N \mid \theta_1, \mathbf{S}_{2:N-1}) = \int g^Q_N(S_N \mid \overline{\theta})\, dF^Q_{\overline{\theta}}(\overline{\theta} \mid \theta_1, \mathbf{S}_{2:N-1}).$$

Since $g^Q_N(S_N \mid \overline{\theta}) = f_{\theta^\perp+\varepsilon}(S_N - \overline{\theta} - a_N)$, which is uniformly bounded by some M for all $(S_N, \overline{\theta})$, $g^Q_N(S_N \mid \theta_1, \mathbf{S}_{2:N-1})$ is uniformly bounded by M as well for all (S_N, θ_1). Further, by Bayes' rule

$$f^Q_{\overline{\theta}}(\overline{\theta} \mid \theta_1, \mathbf{S}_{2:N-1}) = \frac{f^Q_{\theta_1}(\theta_1 \mid \overline{\theta}, \mathbf{S}_{2:N-1}) f_{\overline{\theta}}(\overline{\theta} \mid \mathbf{S}_{2:N-1})}{f^Q_{\theta_1}(\theta_1 \mid \mathbf{S}_{2:N-1})}.$$

Now, θ_1 is independent of $\mathbf{S}_{2:N-1}$ conditional on $\overline{\theta}$, and so $f^Q_{\theta_1}(\theta_1 \mid \overline{\theta}, \mathbf{S}_{2:N-1}) = f^Q_{\theta_1}(\theta_1 \mid \overline{\theta}) = f_{\theta^\perp}(\theta_1 - \overline{\theta})$. Then $f^Q_{\overline{\theta}}(\overline{\theta} \mid \theta_1, \mathbf{S}_{2:N-1})$ is equivalently

$$f^Q_{\overline{\theta}}(\overline{\theta} \mid \theta_1, \mathbf{S}_{2:N-1}) = \frac{f_{\theta^\perp}(\theta_1 - \overline{\theta}) f_{\overline{\theta}}(\overline{\theta} \mid \mathbf{S}_{2:N-1})}{f^Q_{\theta_1}(\theta_1 \mid \mathbf{S}_{2:N-1})}.$$

Inserting this into the previous expression for $g^Q_N(S_N \mid \theta_1, \mathbf{S}_{2:N-1})$ yields

$$g^Q_N(S_N \mid \theta_1, \mathbf{S}_{2:N-1}) = \frac{1}{f^Q_{\theta_1}(\theta_1 \mid \mathbf{S}_{2:N-1})} \int f_{\theta^\perp+\varepsilon}(S_N - \overline{\theta} - a_N) f_{\theta^\perp}(\theta_1 - \overline{\theta})\, dF^Q_{\overline{\theta}}(\overline{\theta} \mid \mathbf{S}_{2:N-1}).$$

Applying Lemma O.3 to an $(N-1)$-agent model implies that $f^Q_{\theta_1}(\theta_1 \mid \mathbf{S}_{2:N-1})$ is continuous in θ_1. Meanwhile by assumption $f_{\theta^\perp+\varepsilon}(S_N - \overline{\theta} - a_N)$ and $f_{\theta^\perp}(\theta_1 - \overline{\theta})$ are both continuous in (S_N, θ_1) for every $\overline{\theta}$, and are uniformly bounded above for every $(\theta_1, S_N, \overline{\theta})$. Then by the dominated convergence theorem the integral is also continuous in (S_N, θ_1), ensuring that $g^Q_N(S_N \mid \theta_1, \mathbf{S}_{2:N-1})$ is a continuous function of (S_N, θ_1). Finally, consider differentiating wrt S_N. As $f'_{\theta^\perp+\varepsilon}$ exists and is uniformly bounded, and $f_{\theta^\perp}$ is also uniformly bounded, the Leibniz integral rule ensures that

$$\frac{\partial}{\partial S_N} g^Q_N(S_N \mid \theta_1, \mathbf{S}_{2:N-1}) = \frac{1}{f^Q_{\theta_1}(\theta_1 \mid \mathbf{S}_{2:N-1})} \int f'_{\theta^\perp+\varepsilon}(S_N - \overline{\theta} - a_N) f_{\theta^\perp}(\theta_1 - \overline{\theta})\, dF^Q_{\overline{\theta}}(\overline{\theta} \mid \mathbf{S}_{2:N-1}).$$

Since $f'_{\theta^\perp+\varepsilon}$ is also continuous, this expression is continuous in (S_N, θ_1) following the same logic which ensured that $g_N^Q(S_N \mid \theta_1, \mathbf{S}_{2:N-1})$ is continuous. Finally, let M be an upper bound on $|f'_{\theta^\perp+\varepsilon}|$. Then as

$$f_{\theta_1}^Q(\theta_1 \mid \mathbf{S}_{2:N-1}) = \int f_{\theta^\perp}(\theta_1 - \overline{\theta})\, dF_{\overline{\theta}}^Q(\overline{\theta} \mid \mathbf{S}_{2:N-1}),$$

it follows that $\left|\frac{\partial}{\partial S_N} g_N^Q(S_N \mid \theta_1, \mathbf{S}_{2:N-1})\right|$ is uniformly bounded above by M as well. So $g_N^Q(S_N \mid \theta_1, \mathbf{S}_{2:N-1})$ satisfies condition 2.

Now consider the circumstance linkage model. In this model

$$g_N^C(S_N \mid \theta_1 = t, S_1 = s, \mathbf{S}_{2:N-1}) = g_N^C(S_N \mid \varepsilon_1 = s - t - a_1, \mathbf{S}_{2:N-1}),$$

as $\varepsilon_1 = S_1 - \theta_1 - a_1$ and S_N is independent of S_1 conditional on ε_1. It is therefore enough to establish that $g_N^C(S_N \mid \varepsilon_1, \mathbf{S}_{2:N-1})$ is a $C^{1,0}$ function of (S_N, ε_1) with uniform bounds on it and its derivative wrt S_N. This follows from work nearly identical to the previous paragraph, substituting ε_1 for θ_1 and $\overline{\varepsilon}$ for $\overline{\theta}$. □

O.2 Proofs for the Gaussian Setting

O.2.1 Verification of Assumptions in 2.6

Here we verify that Gaussian uncertainty satisfies the stated assumptions. Assumptions 1, 2, 3, and 5 are immediate. Assumption 6 is satisfied for any strictly convex cost function, since the second derivative of the posterior expectation in each signal realization is zero. Assumption 4 is verified in the following lemma:

Lemma O.5. *Suppose $\xi \sim \mathcal{N}(0, \sigma^2)$. Then for any $\overline{\Delta} > 0$, the function*

$$J^*(\xi) = \frac{1}{\overline{\Delta}^2}\left(\exp\left(\frac{\overline{\Delta}^2}{2\sigma^2}\right) + \exp\left(\frac{\overline{\Delta}|\xi|}{\sigma^2}\right) - 2\right)^2$$

satisfies $|J(\xi, \Delta)| \le J^*(\xi)$ for every $\xi \in \mathbb{R}$ and $\Delta \in [-\overline{\Delta}, \overline{\Delta}]$, and $\mathbb{E}[J^*(\xi)] < \infty$.

Proof. Under the distributional assumption on ξ, the density function f_ξ has the form

$$f_\xi(\xi) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{\xi^2}{2\sigma^2}\right).$$

Therefore

$$\frac{1}{\Delta}\frac{f_\xi(\xi - \Delta) - f_\xi(\xi)}{f_\xi(\xi)} = \frac{\exp\left(\frac{1}{\sigma^2}\Delta(\xi - \Delta/2)\right) - 1}{\Delta}.$$

Now, we may equivalently write

$$\begin{aligned}\frac{1}{\Delta}\frac{f_\xi(\xi - \Delta) - f_\xi(\xi)}{f_\xi(\xi)} &= \frac{1}{\sigma^2}\int_{\Delta/2}^{\xi} \exp\left(\frac{1}{\sigma^2}\Delta(\widetilde{\xi} - \Delta/2)\right) d\widetilde{\xi} \\ &= \frac{\exp\left(-\frac{\Delta^2}{2\sigma^2}\right)}{\sigma^2}\int_{\Delta/2}^{\xi} \exp\left(\frac{\Delta\widetilde{\xi}}{\sigma^2}\right) d\widetilde{\xi}.\end{aligned}$$

Hence

$$\left|\frac{1}{\Delta}\frac{f_\xi(\xi-\Delta)-f_\xi(\xi)}{f_\xi(\xi)}\right| = \frac{\exp\left(-\frac{\Delta^2}{2\sigma^2}\right)}{\sigma^2}\int_{\min\{\Delta/2,\xi\}}^{\max\{\Delta/2,\xi\}} \exp\left(\frac{\Delta\widetilde{\xi}}{\sigma^2}\right)\, d\widetilde{\xi}$$
$$\leq \frac{1}{\sigma^2}\int_{\min\{\Delta/2,\xi\}}^{\max\{\Delta/2,\xi\}} \exp\left(\frac{\Delta\widetilde{\xi}}{\sigma^2}\right)\, d\widetilde{\xi}.$$

Let

$$H(\xi,\Delta) \equiv \frac{1}{\sigma^2}\int_{\min\{\Delta/2,\xi\}}^{\max\{\Delta/2,\xi\}} \exp\left(\frac{\Delta\widetilde{\xi}}{\sigma^2}\right)\, d\widetilde{\xi}.$$

We will show that $H(\xi,\Delta) \leq \sqrt{J^*(\xi)}$ for all ξ and $\Delta \in [-\overline{\Delta},\overline{\Delta}]$ in cases, depending on the signs of ξ, Δ, and $\xi - \Delta/2$.

Case 1: $\xi \geq \Delta/2 \geq 0$. Then

$$H(\xi,\Delta) = \frac{1}{\sigma^2}\int_{\Delta/2}^{\xi} \exp\left(\frac{\Delta\widetilde{\xi}}{\sigma^2}\right)\, d\widetilde{\xi}$$
$$\leq \frac{1}{\sigma^2}\int_{0}^{\xi} \exp\left(\frac{\overline{\Delta}\widetilde{\xi}}{\sigma^2}\right)\, d\widetilde{\xi} = \frac{1}{\overline{\Delta}}\left(\exp\left(\frac{\overline{\Delta}\xi}{\sigma^2}\right)-1\right) \leq \sqrt{J^*(\xi)}.$$

Case 2: $\xi \geq 0 > \Delta/2$. Then

$$H(\xi,\Delta) = \frac{1}{\sigma^2}\int_{\Delta/2}^{\xi} \exp\left(\frac{\Delta\widetilde{\xi}}{\sigma^2}\right)\, d\widetilde{\xi}$$
$$\leq \frac{1}{\sigma^2}\left(\int_{0}^{\xi} \exp\left(\frac{\overline{\Delta}\widetilde{\xi}}{\sigma^2}\right)\, d\widetilde{\xi} + \int_{-\overline{\Delta}/2}^{0} \exp\left(-\frac{\overline{\Delta}\widetilde{\xi}}{\sigma^2}\right)\, d\widetilde{\xi}\right)$$
$$= \frac{1}{\overline{\Delta}}\left(\exp\left(\frac{\overline{\Delta}\xi}{\sigma^2}\right)+\exp\left(\frac{\overline{\Delta}^2}{2\sigma^2}\right)-2\right) = \sqrt{J^*(\xi)}.$$

Case 3: $\Delta/2 > \xi \geq 0$. Then

$$H(\xi,\Delta) = \frac{1}{\sigma^2}\int_{\xi}^{\Delta/2} \exp\left(\frac{\Delta\widetilde{\xi}}{\sigma^2}\right)\, d\widetilde{\xi}$$
$$\leq \frac{1}{\sigma^2}\int_{0}^{\overline{\Delta}/2} \exp\left(\frac{\overline{\Delta}\widetilde{\xi}}{\sigma^2}\right)\, d\widetilde{\xi} = \frac{1}{\overline{\Delta}}\left(\exp\left(\frac{\overline{\Delta}^2}{2\sigma^2}\right)-1\right) \leq \sqrt{J^*(\xi)}.$$

Case 4: $\Delta/2 > 0 > \xi$. Then

$$H(\xi,\Delta) = \frac{1}{\sigma^2}\int_{\xi}^{\Delta/2} \exp\left(\frac{\Delta\widetilde{\xi}}{\sigma^2}\right)\, d\widetilde{\xi}$$
$$\leq \frac{1}{\sigma^2}\left(\int_{0}^{\overline{\Delta}/2} \exp\left(\frac{\overline{\Delta}\widetilde{\xi}}{\sigma^2}\right)\, d\widetilde{\xi} + \int_{\xi}^{0} \exp\left(-\frac{\overline{\Delta}\widetilde{\xi}}{\sigma^2}\right)\, d\widetilde{\xi}\right)$$
$$= \frac{1}{\overline{\Delta}}\left(\exp\left(\frac{\overline{\Delta}^2}{2\sigma^2}\right)+\exp\left(\frac{\overline{\Delta}|\xi|}{\sigma^2}\right)-2\right) = \sqrt{J^*(\xi)}.$$

Case 5: $0 \geq \Delta/2 > \xi$. Then

$$\begin{aligned} H(\xi,\Delta) &= \frac{1}{\sigma^2}\int_{\xi}^{\Delta/2} \exp\left(\frac{\Delta\widetilde{\xi}}{\sigma^2}\right) d\widetilde{\xi} \\ &\leq \frac{1}{\sigma^2}\int_{\xi}^{0} \exp\left(-\frac{\overline{\Delta}\widetilde{\xi}}{\sigma^2}\right) d\widetilde{\xi} = \frac{1}{\overline{\Delta}}\left(\exp\left(\frac{\overline{\Delta}|\xi|}{\sigma^2}\right)-1\right) \leq \sqrt{J^*(\xi)}. \end{aligned}$$

Case 6: $0 > \xi \geq \Delta/2$. Then

$$\begin{aligned} H(\xi,\Delta) &= \frac{1}{\sigma^2}\int_{\Delta/2}^{\xi} \exp\left(\frac{\Delta\widetilde{\xi}}{\sigma^2}\right) d\widetilde{\xi} \\ &\leq \frac{1}{\sigma^2}\int_{-\overline{\Delta}/2}^{0} \exp\left(-\frac{\overline{\Delta}\widetilde{\xi}}{\sigma^2}\right) d\widetilde{\xi} = \frac{1}{\overline{\Delta}}\left(\exp\left(\frac{\overline{\Delta}^2}{2\sigma^2}\right)-1\right) \leq \sqrt{J^*(\xi)}. \end{aligned}$$

This establishes that $|J(\xi,\Delta)| \leq H(\xi,\Delta)^2 \leq J^*(\xi)$ for every ξ and $\Delta \in [-\overline{\Delta},\overline{\Delta}]$, as desired. It remains only to show that J^* is $\mathcal{P}^0$-integrable. This follows because

$$\begin{aligned} J^*(\xi) &\leq \frac{1}{\overline{\Delta}^2}\left(\exp\left(\frac{\overline{\Delta}^2}{2\sigma^2}\right)+\exp\left(\frac{\overline{\Delta}|\xi|}{\sigma^2}\right)\right)^2 \\ &= \frac{1}{\overline{\Delta}^2}\left(\exp\left(\frac{\overline{\Delta}^2}{\sigma^2}\right)+2\exp\left(\frac{\overline{\Delta}^2}{\sigma^2}\right)\exp\left(\frac{\overline{\Delta}|\xi|}{\sigma^2}\right)+\exp\left(\frac{2\overline{\Delta}|\xi|}{\sigma^2}\right)\right) \\ &= \frac{1}{\overline{\Delta}^2}\left(\exp\left(\frac{\overline{\Delta}^2}{\sigma^2}\right)+2\exp\left(\frac{\overline{\Delta}^2}{\sigma^2}\right)\left(\exp\left(\frac{\overline{\Delta}\xi}{\sigma^2}\right)+\exp\left(-\frac{\overline{\Delta}\xi}{\sigma^2}\right)\right)\right. \\ &\qquad \left.+\exp\left(\frac{2\overline{\Delta}\xi}{\sigma^2}\right)+\exp\left(-\frac{2\overline{\Delta}\xi}{\sigma^2}\right)\right) \end{aligned}$$

The first term is a constant, while each of the remaining terms is proportional to a lognormal random variable. Thus each term has finite mean, and hence so does $J^*(\xi)$. □

O.2.2 Marginal Value of Effort

Consider the quality linkage model, and suppose that agent i chooses effort $a_i = a^* + \Delta$ while all agents $j \neq i$ choose the equilibrium effort level a^*. The principal's posterior belief about $\overline{\theta} + \theta_i^{\perp}$ is independent of $\mathbf{S}_{-i}$ conditional on $\overline{\theta}$. Thus, using standard formulas for updating to normal signals, we can first update the principal's belief about $\overline{\theta}$ to $\overline{\theta} \mid \mathbf{S}_{-i} \sim \mathcal{N}\left(\hat{\mu}_{\overline{\theta}}, \hat{\sigma}_{\overline{\theta}^2}\right)$, where

$$\hat{\mu}_{\overline{\theta}} \equiv \frac{(N-1)\sigma_{\overline{\theta}}^2 \cdot (\overline{S}_{-i} - a^*) + (\sigma_{\theta^\perp}^2 + \sigma_\varepsilon^2)\cdot \mu}{(N-1)\sigma_{\overline{\theta}}^2 + \sigma_{\theta^\perp}^2 + \sigma_\varepsilon^2}, \quad \hat{\sigma}_{\overline{\theta}}^2 \equiv \frac{\sigma_{\overline{\theta}}^2}{(N-1)\sigma_{\overline{\theta}^2} + \sigma_{\theta^\perp}^2 + \sigma_\varepsilon^2}.$$

and $\overline{S}_{-i}$ is the average outcome. The principal's expectation of $\overline{\theta} + \theta_i^{\perp}$ after further updating to S_i is

$$\mathbb{E}(\overline{\theta} + \theta_i^{\perp} \mid \mathbf{S}) = \frac{\sigma_\varepsilon^2}{\hat{\sigma}_{\overline{\theta}^2} + \sigma_{\theta^\perp}^2 + \sigma_\varepsilon^2}\cdot(\overline{S}_{-i} - a^*) + \frac{\hat{\sigma}_{\overline{\theta}^2} + \sigma_{\theta^\perp}^2}{\hat{\sigma}_{\overline{\theta}^2} + \sigma_{\theta^\perp}^2 + \sigma_\varepsilon^2}\cdot(S_i - a^*).$$

Taking an expectation with respect to the agent's prior belief, we have:

$$\mu_N(\Delta) = \mathbb{E}\left[\mathbb{E}(\overline{\theta} + \theta_i^{\perp} \mid S)\right] = \frac{\sigma_\varepsilon^2}{\hat{\sigma}_{\overline{\theta}^2} + \sigma_{\theta^\perp}^2 + \sigma_\varepsilon^2} \cdot \mu + \frac{\hat{\sigma}_{\overline{\theta}^2} + \sigma_{\theta^\perp}^2}{\hat{\sigma}_{\overline{\theta}^2} + \sigma_{\theta^\perp}^2 + \sigma_\varepsilon^2} \cdot (\mu + \Delta)$$
$$= \mu + \frac{\hat{\sigma}_{\overline{\theta}^2} + \sigma_{\theta^\perp}^2}{\hat{\sigma}_{\overline{\theta}^2} + \sigma_{\theta^\perp}^2 + \sigma_\varepsilon^2} \cdot \Delta$$

and the marginal value of effort is

$$\mu_N'(\Delta) = \frac{\hat{\sigma}_{\overline{\theta}^2} + \sigma_{\theta^\perp}^2}{\hat{\sigma}_{\overline{\theta}^2} + \sigma_{\theta^\perp}^2 + \sigma_\varepsilon^2}$$
$$= \left(\frac{\sigma_{\overline{\theta}}^2}{(N-1)\sigma_{\overline{\theta}^2} + \sigma_{\theta^\perp}^2 + \sigma_\varepsilon^2} + \sigma_{\theta^\perp}^2\right) \Big/ \left(\frac{\sigma_{\overline{\theta}}^2}{(N-1)\sigma_{\overline{\theta}^2} + \sigma_{\theta^\perp}^2 + \sigma_\varepsilon^2} + \sigma_{\theta^\perp}^2 + \sigma_\varepsilon^2\right). \tag{O.1}$$

It is straightforward to verify that this expression is independent of Δ, decreasing in N, and converges to $\sigma_{\theta^\perp}^2 / \left(\sigma_{\theta^\perp}^2 + \sigma_\varepsilon^2\right)$ as $N \to \infty$.

Consider now the circumstance linkage model. Using parallel arguments to those above, the principal's posterior belief about the common part of the noise shock $\overline{\varepsilon}$ after updating to $\mathbf{S}_{-i}$ is

$$\overline{\varepsilon} \mid \mathbf{S}_{-i} \sim \mathcal{N}\left(\frac{(N-1)\sigma_{\overline{\varepsilon}}^2}{(N-1)\sigma_{\overline{\varepsilon}}^2 + \sigma_\theta^2 + \sigma_{\varepsilon^\perp}^2} \cdot \left(\overline{S}_{-i} - a^* - \mu\right), \frac{\sigma_{\overline{\varepsilon}}^2(\sigma_{\varepsilon^\perp}^2 + \sigma_\theta^2)}{(N-1)\sigma_{\overline{\varepsilon}}^2 + \sigma_{\varepsilon^\perp}^2 + \sigma_\theta^2}\right) \equiv \mathcal{N}(\eta, \hat{\sigma}_{\overline{\varepsilon}}^2)$$

and the principal's posterior expectation of θ_i after further updating to S_i is

$$\mathbb{E}(\theta_i \mid \mathbf{S}) = \frac{\sigma_\theta^2}{\sigma_\theta^2 + \hat{\sigma}_{\overline{\varepsilon}}^2 + \sigma_{\varepsilon^\perp}^2} \cdot (S_i - \eta) + \frac{\hat{\sigma}_{\overline{\varepsilon}}^2 + \sigma_{\varepsilon^\perp}^2}{\sigma_\theta^2 + \hat{\sigma}_{\overline{\varepsilon}}^2 + \sigma_{\varepsilon^\perp}^2} \cdot \mu$$

Since in the agent's prior, $\mathbb{E}(S_i) = \mu + \Delta$ and $\mathbb{E}(\eta) = 0$, the agent's expectation of the principal's forecast is

$$\mu_N(\Delta) = \mathbb{E}(\theta_i \mid \mathbf{S}) = \mu + \frac{\sigma_\theta^2}{\sigma_\theta^2 + \hat{\sigma}_{\overline{\varepsilon}}^2 + \sigma_{\varepsilon^\perp}^2} \cdot \Delta$$

implying that the marginal value of effort is

$$\mu_N'(\Delta) = \sigma_\theta^2 / (\sigma_\theta^2 + \hat{\sigma}_{\overline{\varepsilon}}^2 + \sigma_\varepsilon^2)$$
$$= \sigma_\theta^2 \Big/ \left(\sigma_\theta^2 + \frac{\sigma_{\overline{\varepsilon}}^2(\sigma_{\varepsilon^\perp}^2 + \sigma_\theta^2)}{(N-1)\sigma_{\overline{\varepsilon}}^2 + \sigma_{\varepsilon^\perp}^2 + \sigma_\theta^2} + \sigma_\varepsilon^2\right) \tag{O.2}$$

This expression is constant in Δ, increasing in N, and converges to $\sigma_\theta^2 / (\sigma_\theta^2 + \sigma_\varepsilon^2)$ as N grows large.

O.3 Proofs for Section 6 (Extensions)

O.3.1 Proof of Proposition 4

Consider first the quality linkage model. Let $\mu_m(\Delta)$ be the agent's value of distortion when $m \in \{0, ...J\}$ linkages have been identified. As in the main model, this value is differentiable and independent of the action the principal expects the agent to take. (See the proof of Lemma C.1.) Agent 0's equilibrium effort is then determined by

$$\mu_m'(0) = C'(a_0).$$

We prove that $\mu_m'(0) > \mu_{m+1}'(0)$ for every m.

Let $\mathbf{S}^j = (S_1^j, ..., S_{N_j}^j)$ be the vector of signal realizations for each segment j, and $\mathbf{S}^{1:m}$ for the matrix of signal realizations for all signal realizations from segments 1 through m. We will write G^j for the distribution function of each $\mathbf{S}^j$, and $G^{0:m}$ for the distribution function of $(S^0, \mathbf{S}^{1:m})$. Dropping explicit conditioning on actions for convenience, a change of variables as in the proof of Lemma 1 allows us to write $\mu_m(\Delta)$ and $\mu_{m+1}(\Delta)$ as

$$\mu_m(\Delta) = \int dG^{0:m}(S_0 = s_0, \mathbf{S}^{1:m})\, \mathbb{E}[\theta_0 \mid S_0 = s_0 + \Delta, \mathbf{S}^{1:m}]$$

and

$$\mu_{m+1}(\Delta) = \int dG^{0:m+1}(S_0 = s_0, \mathbf{S}^{1:m+1})\, \mathbb{E}[\theta_0 \mid S_0 = s_0 + \Delta, \mathbf{S}^{1:m+1}]$$

for some common set of actions. The law of iterated expectations applied to $\mathbb{E}[\theta_0 \mid S_0 = s_0 + \Delta, \mathbf{S}^{1:m}]$ allows the previous expression for $\mu_m(\Delta)$ to be expanded as

$$\begin{aligned}\mu_m(\Delta) = &\int dG^{0:m}(S_0 = s_0, \mathbf{S}^{1:m}) \\ &\times \int dG^{m+1}(\mathbf{S}^{m+1} \mid S_0 = s_0 + \Delta, \mathbf{S}^{0:m})\, \mathbb{E}[\theta_0 \mid S_0 = s_0 + \Delta, \mathbf{S}^{1:m+1}].\end{aligned}$$

Meanwhile the law of iterated expectations applied to the outer expectation allows $\mu_{m+1}(\Delta)$ to be expanded as

$$\begin{aligned}\mu_{m+1}(\Delta) = &\int dG^{0:m}(S_0 = s_0, \mathbf{S}^{1:m}) \\ &\times \int dG^{m+1}(\mathbf{S}^{m+1} \mid S_0 = s_0, \mathbf{S}^{0:m})\, \mathbb{E}[\theta_0 \mid S_0 = s_0 + \Delta, \mathbf{S}^{1:m+1}].\end{aligned}$$

Each of these inner integrals may be further expanded using the law of total probability, yielding

$$\begin{aligned}\mu_m(\Delta) = &\int dG^{0:m}(S_0 = s_0, \mathbf{S}^{1:m}) \\ &\times \int dF_{\overline{\theta}^{m+1}}(\overline{\theta}^{m+1} \mid S_0 = s_0 + \Delta, \mathbf{S}^{1:m}) \\ &\times \int dG^{m+1}(\mathbf{S}^{m+1} \mid \overline{\theta}^{m+1})\, \mathbb{E}[\theta_0 \mid S_0 = s_0 + \Delta, \mathbf{S}^{1:m+1}]\end{aligned}$$

and

$$\begin{aligned}\mu_{m+1}(\Delta) = &\int dG^{0:m}(S_0 = s_0, \mathbf{S}^{1:m}) \\ &\times \int dF_{\overline{\theta}^{m+1}}(\overline{\theta}_{m+1} \mid S_0 = s_0, \mathbf{S}^{1:m}) \\ &\times \int dG^{m+1}(\mathbf{S}^{m+1} \mid \overline{\theta}^{m+1})\, \mathbb{E}[\theta_0 \mid S_0 = s_0 + \Delta, \mathbf{S}^{1:m+1}]\end{aligned}$$

where we have used the fact that $\mathbf{S}^{m+1}$ is independent of $(S_0, \mathbf{S}^{1:m})$ conditional on $\overline{\theta}_{m+1}$ to drop extraneous conditioning in the inner expectation.

So define a function $\psi(\delta_1, \delta_2, s_0, \mathbf{S}^{1:m})$ by

$$\psi(\delta_1, \delta_2, s_0, \mathbf{S}^{1:m}) \equiv \int dF_{\overline{\theta}^{m+1}}(\overline{\theta}^{m+1} \mid S_0 = s_0 + \delta_1, \mathbf{S}^{1:m}) \times \int dG^{m+1}(\mathbf{S}^{m+1} \mid \overline{\theta}^{m+1})\, \mathbb{E}[\theta_0 \mid S_0 = s_0 + \delta_2, \mathbf{S}^{1:m+1}].$$

Then for every $\Delta > 0$ we have

$$\frac{1}{\Delta}\mu_m(\Delta) - \mu_{m+1}(\Delta) = \int dG^{0:m}(S_0 = s_0, \mathbf{S}^{1:m}) \, \frac{1}{\Delta} \left(\psi(\Delta, \Delta, s_0, \mathbf{S}^{1:m}) - \psi(\Delta, 0, s_0, \mathbf{S}^{1:m}) \right).$$

Since

$$\mu_m'(0) - \mu_{m+1}'(0) = \lim_{\Delta \downarrow 0}(\mu_m(\Delta) - \mu) - \lim_{\Delta \downarrow 0}(\mu_{m+1}(\Delta) - \mu) = \lim_{\Delta \downarrow 0}(\mu_m(\Delta) - \mu_{m+1}(\Delta),$$

It is therefore sufficient to determine the limiting behavior of

$$\frac{1}{\Delta} \left(\psi(\Delta, \Delta, s_0, \mathbf{S}^{1:m}) - \psi(\Delta, 0, s_0, \mathbf{S}^{1:m}) \right)$$

as $\Delta \downarrow 0$.

Note that

$$S_i^{m+1} = \overline{\theta}^m + \theta_i^{\perp, m} + \varepsilon_i,$$

where the densities of $\theta_i^{\perp, m}$ and ε_i each exist and are bounded by assumption. Then there exists a differentiable distribution function H with bounded derivative such that $G_i^{m+1}(S_i^{m+1} \mid \overline{\theta}^{m+1}) = H(S_i^{m+1} - \overline{\theta}^{m+1})$ for each agent i in segment $m+1$. Since the elements of $\mathbf{S}^{m+1}$ are independent conditional on $\overline{\theta}^{m+1}$, we may write

$$G^{m+1}(\mathbf{S}^{m+1} \mid \theta^{m+1}) = \prod_{i=1}^{N_{m+1}} H(S_i^{m+1} - \overline{\theta}^{m+1}).$$

A change of variables therefore yields

$$\int dG^{m+1}(\mathbf{S}^{m+1} \mid \overline{\theta}^{m+1})\, \mathbb{E}[\theta_0 \mid S_0 = s_0 + \delta_1, \mathbf{S}^{1:m+1}]$$
$$= \int_0^1 dq_1 ... \int_0^1 dq_{N_{m+1}} \mathbb{E}[\theta_0 \mid S_0 = s_0 + \delta_2, \mathbf{S}^{1:m}, \mathbf{S}^{m+1} = (H^{-1}(q_i) + \overline{\theta}^{m+1})_{i=1...,N_m}].$$

Now fix s_0 and $\mathbf{S}^{1:m}$, and denote the integrand of this representation

$$\zeta(z, \delta, \mathbf{q}) \equiv \mathbb{E}[\theta_0 \mid S_0 = s_0 + \delta, \mathbf{S}^{1:m}, \mathbf{S}^{m+1} = (H^{-1}(q_i) + z)_{i=1...,N_m}],$$

where $\mathbf{q} \equiv (q_1, ..., q_{N_{m+1}})$. Using techniques very similar to that used to prove Lemma B.3, it can be shown that there exists a C^1 quantile function $\phi(q, \delta)$ satisfying $\partial \phi / \partial \delta > 0$ such that

$F_{\overline{\theta}^{m+1}}(\phi(q,\delta) \mid S_0 = s_0 + \delta, \mathbf{S}^{1:m}) = q$ for every q_0 and Δ. Then by a further change of variables, ψ may be written

$$\psi(\delta_1, \delta_2, s_0, \mathbf{S}^{1:m}) = \int_0^1 dq_0 ... \int_0^1 dq_{N_{m+1}}\, \zeta(\phi(q_0, \delta_1), \delta_2, \mathbf{q}).$$

By assumption, $\mathbb{E}[\theta_0 \mid S_0, \mathbf{S}^{1:m+1}]$ is differentiable wrt each S_i^{m+1}, and by arguments very similar to those used to prove Lemma B.8, it can be shown that $\frac{\partial}{\partial S_i^{m+1}} \mathbb{E}[\theta_0 \mid S_0, \mathbf{S}^{1:m+1}] > 0$ for every $i = 1, ..., m+1$. Since additionally $\partial\phi/\partial\delta > 0$ everywhere, it follows that

$$\zeta(\phi(q_0, \Delta), \Delta, \mathbf{q}) > \zeta(\phi(q_0, 0), \Delta, \mathbf{q})$$

for every $\Delta > 0$ and $(q_0, \mathbf{q})$, and thus that

$$\begin{aligned} &\frac{1}{\Delta}\left(\psi(\Delta, \Delta, s_0, \mathbf{S}^{1:m}) - \psi(\Delta, 0, s_0, \mathbf{S}^{1:m})\right) \\ &= \int_0^1 dq_0 ... \int_0^1 dq_{N_{m+1}}\, \frac{1}{\Delta}(\zeta(\phi(q_0, \Delta), \Delta, \mathbf{q}) - \zeta(\phi(q_0, 0), \Delta, \mathbf{q})) \end{aligned}$$

is strictly positive for every $\Delta > 0$. Since this result holds for every $(s_0, \mathbf{S}^{1:N})$, Fatou's lemma therefore implies that

$$\mu_m'(0) - \mu_{m+1}'(0) \geq \int dG^{0:m}(S_0 = s_0, \mathbf{S}^{1:m}) \liminf_{\Delta \downarrow 0} \frac{1}{\Delta}\left(\psi(\Delta, \Delta, s_0, \mathbf{S}^{1:m}) - \psi(\Delta, 0, s_0, \mathbf{S}^{1:m})\right)$$

and

$$\begin{aligned} &\liminf_{\Delta \downarrow 0} \frac{1}{\Delta}\left(\psi(\Delta, \Delta, s_0, \mathbf{S}^{1:m}) - \psi(\Delta, 0, s_0, \mathbf{S}^{1:m})\right) \\ &\geq \int_0^1 dq_0 ... \int_0^1 dq_{N_{m+1}} \lim_{\Delta \downarrow 0} \frac{1}{\Delta}(\zeta(\phi(q_0, \Delta), \Delta, \mathbf{q}) - \zeta(\phi(q_0, 0), \Delta, \mathbf{q})). \end{aligned}$$

Further, the integrand of the previous expression can be equivalently written

$$\begin{aligned} &\frac{1}{\Delta}(\zeta(\phi(q_0, \Delta), \Delta, \mathbf{q}) - \zeta(\phi(q_0, 0), \Delta, \mathbf{q})) \\ &= \frac{1}{\Delta}(\zeta(\phi(q_0, \Delta), \Delta, \mathbf{q}) - \zeta(\phi(q_0, 0), 0, \mathbf{q})) - \frac{1}{\Delta}(\zeta(\phi(q_0, 0), \Delta, \mathbf{q}) - \zeta(\phi(q_0, 0), 0, \mathbf{q})). \end{aligned}$$

Now, by assumption $\mathbb{E}[\theta_0 \mid S_0, \mathbf{S}^{1:m+1}]$ is differentiable wrt S_0 and each S_i^{m+1}, and each derivative is continuous in $(S_0, \mathbf{S}^{m+1})$. Hence $\mathbb{E}[\theta_0 \mid S_0, \mathbf{S}^{1:m+1}]$ is a totally differentiable function of $(S_0, \mathbf{S}^{1:m+1})$ everywhere. Thus by the chain rule

$$\begin{aligned} &\lim_{\Delta \downarrow 0} \frac{1}{\Delta}(\zeta(\phi(q_0, \Delta), \Delta, \mathbf{q}) - \zeta(\phi(q_0, 0), \Delta, \mathbf{q})) \\ &= \frac{\partial}{\partial S_0} \mathbb{E}[\theta_0 \mid S_0 = s_0, \mathbf{S}^{1:m}, \mathbf{S}^{m+1} = (H^{-1}(q_i) + \phi(q_0, 0))_{i=1...,N_m}] \\ &\quad + \sum_{i=1}^{N_m} \frac{\partial}{\partial S_i^{m+1}} \mathbb{E}[\theta_0 \mid S_0 = s_0, \mathbf{S}^{1:m}, \mathbf{S}^{m+1} = (H^{-1}(q_i) + \phi(q_0, 0))_{i=1...,N_m}] \frac{\partial \phi}{\partial \Delta}(q_0, 0) \\ &\quad - \frac{\partial}{\partial S_0} \mathbb{E}[\theta_0 \mid S_0 = s_0, \mathbf{S}^{1:m}, \mathbf{S}^{m+1} = (H^{-1}(q_i) + \phi(q_0, 0))_{i=1...,N_m}] \\ &= \sum_{i=1}^{N_m} \frac{\partial}{\partial S_i^{m+1}} \mathbb{E}[\theta_0 \mid S_0 = s_0, \mathbf{S}^{1:m}, \mathbf{S}^{m+1} = (H^{-1}(q_i) + \phi(q_0, 0))_{i=1...,N_m}] \frac{\partial \phi}{\partial \Delta}(q_0, 0). \end{aligned}$$

As noted earlier, each of these derivatives is strictly positive, and so it follows that the entire limit is strictly positive. Thus

$$\liminf_{\Delta \downarrow 0} \frac{1}{\Delta} \left(\psi(\Delta, \Delta, s_0, \mathbf{S}^{1:m}) - \psi(\Delta, 0, s_0, \mathbf{S}^{1:m}) \right) > 0$$

everywhere, meaning in turn that $\mu'_m(0) - \mu'_{m+1}(0) > 0$. In other words, the marginal value of effort is declining in m in the quality linkage model.

The result for the circumstance linkage model proceeds nearly identically, with the key difference that now an analog of Lemma B.8 implies that $\frac{\partial}{\partial S_i^{m+1}} \mathbb{E}[\theta_0 \mid S_0, \mathbf{S}^{1:m+1}] < 0$ for every i. Thus

$$\lim_{\Delta \downarrow 0} \frac{1}{\Delta} (\zeta(\phi(q_0, 0), \Delta, \mathbf{q}) - \zeta(\phi(q_0, \Delta), \Delta, \mathbf{q})) > 0$$

everywhere, so that

$$\liminf_{\Delta \downarrow 0} \frac{1}{\Delta} \left(\psi(\Delta, 0, s_0, \mathbf{S}^{1:m}) - \psi(\Delta, \Delta, s_0, \mathbf{S}^{1:m}) \right) > 0$$

everywhere and hence $\mu'_{m+1}(0) - \mu'_m(0) > 0$. So the marginal value of effort is rising in m in the circumstance linkage model.