Data Linkages and Incentives*

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Abstract

Many organizations, such as banks and insurers, determine what services to offer based on a perceived quality of the recipient, e.g. their creditworthiness. Increasingly, organizations have access to new data about consumers, such as categorizations into demographic and lifestyle segments. When organizations learn about a consumer's quality from the behavior of other consumers in the same segment—creating *data linkages*—what are the consequences for each consumer's incentives to exert effort, e.g. to maintain a good credit rating? We study a multiple-agent career concerns model in which agents choose whether to interact with a principal and how much costly effort to exert. Data linkages create informational externalities across consumers, shaping participation rates and effort provision in equilibrium. We show that whether these are welfare-improving depends crucially on whether linkages are about quality (revealing correlations in underlying types) or about a shared circumstance (helping the principal to de-bias shared shocks to observed outcomes).

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1 Introduction

A bank receives a credit card application from a consumer, Alice. The bank observes a small set of traditional covariates about Alice, including her past repayment history and her mix of credit cards and other loans. The bank also learns from a data broker that Alice has been classified as charitable giver, based on donor lists acquired from various charities and Alice's social media activities. The bank notices that other individuals in the "charitable giver" segment have a high rate of credit repayment on average, and provides Alice with a high credit limit.

This narrative is fictional, but it is reminiscent of emerging practices in industry. Data brokers regularly aggregate personal data about consumers, and use this data to identify segments of consumers with similar characteristics and likely behaviors. The range of consumer segments is diverse (see Appendix A for a list). Some segments focus on lifestyle similarities, e.g. "Bible Lifestyle," "Soccer Mom," "Exercise—Sporty Living," "New Age/Organic Lifestyle." Others group consumers based on preferences or activities, e.g. "Outdoor/Hunting & Shooting," "Leans Left," or "Charitable Giving." Still others focus on recent life events, such as getting married, buying a home, or sending a child to college. These segmentations are passed onto companies such as banks and insurance agencies; increasingly, they are being used for decisions regarding what level of service to provide a consumer.[1]

Identifying similarities across consumers can help an organization to better predict their behaviors. But categorization also reshapes incentives for effort, e.g. building credit and driving more attentively. If Alice knows that the bank evaluates her not only based on her own repayment history, but also on the repayment histories of other individuals in her category, does that attenuate her incentives to exercise financial prudence? Since effort in such contexts can be socially valuable, it is important to understand the externalities created

[1] In 2008, the subprime lender CompuCredit was revealed to have reduced credit lines based on visits to various "red flag" establishments, including marriage counselors and nightclubs. (See: https://www.bloomberg.com/news/articles/2008-06-18/your-lifestyle-may-hurt-your-credit.) Some health insurance companies acquire predictions from data brokers like LexisNexis for anticipated health (e.g. the data broker might use whether a woman has recently changed her name to predict pregnancy). (See: https://www.pbs.org/newshour/health/why-health-insurers-track-when-you-buy-plus-size-clothes-or-binge-watch-tv.) And the car insurance company Allstate recently filed a patent for adjusting insurance rates based on routes and historical accident patterns. (See: https://www.usatoday.com/story/money/personalfinance/2016/11/14/route-risk-patent–car-insurance-rate-price/93287372.)

when organizations use data from one individual to inform predictions about others.

To study this question, we focus on one consumer segment and build a model of the incentives of the agents within it. Each agent has an unknown *characteristic* or *type* (e.g., creditworthiness), which a principal (a bank) would like to predict. Agents choose whether to opt-in to interaction with the principal (sign up for a credit card). The principal observes an outcome (the agent's past repayment behavior) from each agent who opts in, which is informative about the agent's underlying type, but also perhaps about the types of others. The agent can manipulate his or her own outcome via costly effort (by being more financially prudent). We say that a *data linkage* exists when a principal bases its prediction of the agent's type on the outcomes of other participating agents, in addition to the agent's own data. Data linkages create an informational externality across agents. Formally, our framework is a multiple-agent version of the classic career concerns model (Holmstrom, 1982), where signals are correlated across agents.

It is useful to distinguish two ways in which agents' outcomes may be informative about one another. The first type of data linkage relates to *quality.* It corresponds to a situation in which, within a segment, the types that the principal cares about forecasting are correlated. In the credit example, this relationship may be a lifestyle habit (e.g. "Frequent Flier," "Charitable Giving," "Exercise—Sporty Living") or personal characteristic (e.g. "Working-class Mom," "Spanish Speaker"). Lending outcomes from consumers in such a segment can be used to better predict repayment for other similar individuals. The second case is a data linkage relating to a common *circumstance.* It corresponds to correlated shocks within a segment. For example, people who commute on the same roads to work are all exposed to the same shocks in local road conditions—e.g. due to construction or bad weather—and an auto insurer will want to take that into account insofar as it helps the insurer to discern the residual differences in the quality of people's driving.

We study how these different types of data linkages affect both consumers' willinginess to opt in to relationships with a principal, as well as their incentives to exert effort in the relationship. Our main results show that these two models of data linkages result in starkly different equilibrium outcomes. When linkages across individuals in a segment are about quality, individuals fully opt-in to interaction with the organization, but exert *low* effort after doing so. In contrast, when the identified linkage is about circumstances, individuals participate with the organization at low rates, but exert *high* effort conditional on doing so. These results are robust to uncertainty about the details of the segment of which the agent is a part, such as the strength of the correlation between outcomes and the size of the population. They rely only on agents' understanding of whether their linkage to other agents regards quality or circumstance.

The main intuition is as follows. In the linked quality model, consumer data are *substitutable*—for instance, observation of repayment rates for other charitable givers helps a bank to learn an average type for this segment, reducing the informativeness of any given borrower's outcome. Thus, distortion of one's own outcome via exerting extra effort will have a smaller influence on the principal's perception about one's own type once consumers are linked. In contrast, in the linked circumstance model, consumer data are *complementary*—for instance, observation of accident rates for other drivers who take the same roads to work is informative about the size of the "road condition shock" on insurance claims. Understanding the size of this shock can help an auto insurer to de-bias outcomes observed for other drivers on those roads. This makes each agent's own outcome more informative about his type, increasing the incentive to exert effort.

Formally, we show that when a data linkage relates to quality, then the marginal value of exerting additional effort is monotonically *decreasing* in the number of agents (vanishing to zero if types are perfectly correlated, and to a strictly positive limit otherwise). When the linkage relates to circumstance, the equilibrium effort level is monotonically *increasing* (with a finite limit). These comparative statics in the marginal value of effort imply the same comparative statics for the equilibrium level of effort exerted by consumers.

The main difficulty in demonstrating this is that the principal's posterior expectation is generally a complex nonlinear relationship that cannot be expressed in closed-form. (An important exception is the case of Gaussian unknowns.) To study equilibrium outcomes, we establish comparative statics of the principal's expected forecast as the distribution of outcomes shifts, drawing on general statistical properties of inference from noisy signals. We also prove a new bound on the sensitivity of the principal's posterior's expectation to the realization of a single signal (when updating from multiple signals), allowing us to characterize its asymptotic sensitivity to information as the number of auxiliary signals grows large. This result is of independent interest, relating to recent work in Kartik, Lee and Suen (2019) and the classic merging results of Blackwell and Dubins (1962).

We next turn to participation. Our above results imply that agent participation decisions in the linked quality model are strategic complements: Opt-in decisions improve the payoffs to opting-in for other agents, and so there is a unique equilibrium in which all agents choose to participate in the interaction with the principal. In contrast, when linkages are over circumstance, participation creates a negative externality on other agents, and for large populations only partial entry is supported in equilibrium.

This model thus delivers the following prediction about the consequences of data linkages:

If a credit card issuer links borrowers in a segment, where individuals in that segment share a linkage about quality—e.g. a common lifestyle or level of financial literacy—individuals will retain their credit cards, but this linkage will cause individuals in that segment to exert lower effort. An organization that values effort may therefore prefer to commit to *not* using big data analytics for forecasting agent characteristics.

On the other hand, if individuals within the segment share a common circumstance—e.g. having recently moved or started a new job—then the linkage induces participating agents to exert higher effort. Depending on the size of the segment, these linkages may also cause some individuals to withdraw from interaction with the principal, e.g. by canceling their credit cards. For small segments, the organization should expect full participation and higher effort, so data linkages unambiguously benefit the principal. On the other hand, if the segment is sufficiently large, then the linkages result in higher effort but lower rates of participation. Whether the organization benefits from use of data linkages then depends on how it trades off between these two goals.

Our results on effort and participation have important implications for the impact of data linkages on social welfare. We show that in both models and for all segment sizes, equilibrium actions are inefficient relative to the first-best (extending a result established in Holmstrom (1982) for Gaussian signals). We additionally compare equilibrium against a "no data linkages" benchmark corresponding to equilibrium when the principal is only permitted to use an agent's *own* past data to predict that agent's type. When agents are connected by a quality linkage, aggregation of data across agents *always* leads to a reduction in social welfare. In contrast, when agents share common circumstances, the welfare implications of data linkages depend on the number of agents within the segment, and can go either way. These results suggest that the type of data being used to link agents is a crucial determinant of the welfare effect of data linkages.

Our paper contributes to an emerging literature regarding the welfare consequences of data markets and algorithmic scoring. This literature has tackled several important social questions, such as whether predictive algorithms discriminate (Chouldechova, 2017; Kleinberg, Mullainathan and Raghavan, 2017; Kearns et al., 2018); how to protect consumers from loss of privacy (Acquisiti, Brandimarte and Loewenstein, 2015; Dwork and Roth, 2014; Fainmesser, Galeotti and Momot, 2019; Eilat, Eliaz and Mu, 2019); how to price data (Bergemann, Bonatti and Smolin, 2018; Agarwal, Dahleh and Sarkar, 2019); whether seller or advertiser access to big data harms consumers (Jullien, Lefouili and Riordan, 2018; Gomes and Pavan, 2018); and how to aggregate big data into market segments or consumer scores (Ichihashi, 2019; Bonatti and Cisternas, 2019; Yang, 2019; Hiri and Vellodi, 2019; Elliot and

Galeotti, 2019). There is additionally a growing literature about strategic interactions with machine learning algorithms: see Eliaz and Spiegler (2018) on the incentives to truthfully report characteristics to a machine learning algorithm, and Olea et al. (2018) on how economic markets select certain models for making predictions over others.

In particular, Acemoglu et al. (2019) and Bergemann, Bonatti and Gan (2019) also consider externalities created by social data. These papers study data sharing in environments where consumers may sell their data. In Bergemann, Bonatti and Gan (2019), other agents' information allows a firm to set more tailored (and possibly personalized) prices, which can decrease consumer surplus. In Acemoglu et al. (2019), agents value privacy, and thus information collected about one agent imposes a direct negative externality on other agents when types are correlated. The externality of interest in the present paper is how information provided by other agents reshapes incentives to exert costly *effort*. As we show, this externality could be positive or negative—in particular, when agents are connected by a quality ilnkage, their equilibrium payoffs are *increasing* in other agents' participation.

At a theoretical level, our paper builds on the career concerns model of Holmstrom (1982). The literature following Holmstrom (1982) has largely focused on signal extraction about a single agent's type in dynamic settings,[2] while we are interested in the externalities of social data in a multiple-agent setting. Our paper is most closely related to Dewatripont, Jewitt and Tirole (1999), which studies how auxiliary data impacts agents incentives for effort. Dewatripont, Jewitt and Tirole (1999) consider the externality of a single auxiliary signal, while we endogenize the auxiliary data as information from other players, who strategically decide whether or not to provide data. Thus, the number of auxiliary signals is determined in equilibrium, and may also be uncertain; this requires comparison of equilibrium actions across various information structures.

Finally, our paper contributes to work on strategic manipulation of information. Recent papers in this category include Frankel and Kartik (2019) and Ball (2019), which characterize the degree to which a principal with commitment power should link his decision to a manipulated signal about the agent's type; Hu, Immorlica and Vaughan (2019), which shows that heterogeneous manipulation costs across different social groups can lead to inequities in outcomes; and Georgiadis and Powell (2019), which studies optimal information acquisition for a designer setting a wage contract. Our paper contributes to this literature by exploring the role of correlations across data for an individual's incentives to manipulate.

[2] A small set of papers, e.g. Auriol, Friebel and Pechlivanos (2002), study career concerns in a multiple agent setting. These papers typically look at effort externalities instead of informational externalities.

2 Model

A single principal interacts with a population of $N < \infty$ agents. Each agent i has a type $\theta_i \in \mathbb{R}$, which is unobserved by all parties and is commonly believed to be drawn from the distribution F_θ with mean $\mu > 0$ and finite variance $\sigma_\theta^2 > 0$. Types are drawn symmetrically but *not* necessarily independently across agents. (We postpone the description of correlation across types until Section 2.3.)

Our setting builds on the classic career concerns model of Holmstrom (1982): each agent's payoffs are increasing in the principal's perception of his type, and the agent can exert costly effort to influence an outcome realization that the principal observes (Section 2.2). Different from Holmstrom (1982), we introduce a preliminary stage at which the agent first chooses whether to opt-in or out of interaction with the principal (Section 2.1), and—most importantly—allow the principal to aggregate the outcomes of multiple agents for prediction. This aggregation is described in detail in Section 2.3.

The model unfolds over three periods, with opt-in/out decisions made in period $t = 0$, effort exerted in period $t = 1$, and forecasts of each agent's type based on outcomes updated in period $t = 2$.

2.1 Period 0. Opt-In/Opt-Out

At period $t = 0$, each agent i first chooses whether to *opt-in* (I) or *opt-out* (O) of an interaction with the principal, where this decision is observed by the principal, but not by other agents. Opting out yields a payoff that we normalize to zero. Below, we use $\mathscr{I}_{\text{opt-in}} \subseteq \{1, \ldots, N\}$ to denote the set of agents who opt-in.

2.2 Period 1. Choice of Costly Effort to Influence Outcome

In period $t = 1$, each agent $i \in \mathscr{I}_{\text{opt-in}}$ chooses a costly effort level $a_i \in \mathbb{R}_+$ to influence an observable outcome S_i, which is related to the agent's type and effort via

$$S_i = \theta_i + a_i + \varepsilon_i,$$

where $\varepsilon_i \sim F_\varepsilon$ is a noise shock satisfying $\mathbb{E}[\varepsilon_i] = 0$ and $\mathbb{E}[\varepsilon_i^2] = \sigma_\varepsilon^2 \in \mathbb{R}_{++}$. Noise shocks are drawn symmetrically but *not* necessarily independently across agents. (We describe the correlation structure across shocks in Section 2.3.) The agent's payoff in this period is

$$R - C(a_i)$$

where $R > 0$ is an opt-in reward, and $C(a_i)$ is the cost to choosing effort a_i.[3] We suppose that the cost function is twice continuously differentiable and satisfies $\lim_{a_i \to \infty} C'(a_i) = \infty$, $C(0) = C'(0) = 0$, and $C''(a_i) > 0$ for all a_i.

2.3 Period 2. Principal's Forecast of Agent's Type

In a second (and final) period, each agent $i \in \mathscr{I}_{\text{opt-in}}$ receives as a payoff the principal's forecast of the agent's type θ_i. The principal's forecast is based on the observed outcomes of all agents who have opted-in; thus, agent i's payoff in the second period is

$$\mathbb{E}\left[\theta_i \mid S_j, j \in \mathscr{I}_{\text{opt-in}}\right]. \tag{1}$$

The agent's total payoff is the sum of his expected payoffs across the two periods.

We contrast below the cases in which agents within the segment are connected by a linkage regarding quality versus circumstance:

Linked Quality. Suppose first that the underlying characteristics help the principal to identify linkages about agent quality. Allowing for imperfect correlation between types, we model this by decomposing θ_i as

$$\theta_i = \overline{\theta} + \theta_i^{\perp},$$

where $\overline{\theta} \sim F_{\overline{\theta}}$ is a common component to the type and $\theta_i^{\perp} \sim F_{\theta^{\perp}}$ is a personal or idiosyncratic component, with each $\theta_i^{\perp}$ independent of $\overline{\theta}$ and all $\theta_j^{\perp}$ for $j \neq i$. Without loss, we assume $\mathbb{E}[\overline{\theta}] = \mu$ while $\mathbb{E}[\theta_i^{\perp}] = 0$. In contrast, the shocks $(\varepsilon_1, ..., \varepsilon_N)$ are mutually independent.

Linked Circumstance. Another possibility is that the characteristics help to identify a shared shock to outcomes, but agent types are not intrinsically related. Formally, we suppose that the noise shock can be decomposed as

$$\varepsilon_i = \overline{\varepsilon} + \varepsilon_i^{\perp}$$

where $\overline{\varepsilon} \sim F_{\overline{\varepsilon}}$ is shared across agents and $\varepsilon_i^{\perp} \sim F_{\varepsilon^{\perp}}$ is idiosyncratic, where each $\varepsilon_i^{\perp}$ is independent of $\overline{\varepsilon}$ and all $\varepsilon_j^{\perp}$ for $j \neq i$. In contrast, agents' types θ_i are mutually independent.

[3] For example, some car insurance companies offer drivers a discount on their insurance premium in return for installing a tracking device (Jin and Vasserman, 2019). In China, opting-in to certain social credit score systems gives users a wide range of benefits including qualification for personal credit loans, preferential treatment at hospitals, and fast-tracked visa applications (Kostka, 2019).

Note that the marginal distribution of each agent's outcome is the same in the two models, the difference is only in the correlation structure across agent outcomes. We will show in our subsequent results that these two kinds of linkages have very different implications fo equilibrium behavior.

2.4 Examples

Commuters and auto-insurers. The principal is an auto-insurer and the agents are commuters. Agent i's type θ_i is his risk level while driving, with higher-type commuters experiencing a lower risk of accidents from driving to work, e.g. because they are more skilled drivers or because their commute takes place over safer roads. Each agent decides whether to own a car versus commuting via rideshares or public transit. Conditional on owning a car, the agent then chooses how much effort to exert towards driving safely, and the insurance company observes his claims rate during this period.

Linkages over quality relevant to the driver's risk level include for example whether a commuter's route to work is primarily via surface streets or highways, which could be discovered based on geolocational data. Linkages over circumstance include whether the driver recently started a new job or was pregnant during the initial claims cycle, as inferred from purchase histories and social media posts.[4]

Consumers and credit-card issuers. The principal is a bank issuing a credit card and agents are consumers. Agent i's type θ_i is a consumer's creditworthiness, with more creditworthy consumers being better able to pay back short-term loans. Each agent decides whether to sign up for a credit card versus paying by debit card or cash. If an agent signs up for a credit card, the agent decides how much effort to exert in order to ensure repayment (e.g. by increasing income or avoiding activities that risk financial loss), and the card issuer observes his repayment rate during the first period.

Linkages over quality relevant to creditworthiness include lifestyle habits and readership of financial and investment news, categories which could be revealed by social media usage and online subscription databases. Linkages over circumstances include whether a consumer's child is currently attending college and whether a family member is currently experiencing a serious illness, as inferred for example from purchasing and travel histories.

[4] A data analyst at Target "was able to identify about 25 products that, when analyzed together, allowed him to assign each shopper a 'pregnancy prediction' score. More important, he could also estimate her due date to within a small window." See: `https://www.nytimes.com/2012/02/19/magazine/shopping-habits.html`.

2.5 Distributional Assumptions

We impose several regularity conditions on the distributions $F_{\overline{\theta}}$, $F_{\theta^\perp}$, $F_{\overline{\varepsilon}}$, and $F_{\varepsilon^\perp}$, which are maintained throughout the paper. Assumptions 1 through 3 are purely technical, and ensure that all distributions have full support and are smooth enough for appropriate derivatives and conditional expectations to exist. Assumptions 4 and 5 are substantive, and ensure monotonicity of inferences about latent variables in outcome and the sufficiency of the first-order approach.

Assumption 1. *The distribution functions $F_{\overline{\theta}}$, $F_{\theta^\perp}$, $F_{\overline{\varepsilon}}$, and $F_{\varepsilon^\perp}$ admit strictly positive, C^1 density functions $f_{\overline{\theta}}$, $f_{\theta^\perp}$, $f_{\overline{\varepsilon}}$, $f_{\varepsilon^\perp}$ with bounded first derivatives on $\mathbb{R}$.*

We will write F_θ and F_ε to denote the distributions of θ_i and ε_i, respectively, with accompanying densities f_θ and f_ε. As these densities are convolutions of densities satisfying the properties of Assumption 1, they satisfy the same properties.

Assumption 2. *For every population size N, model $M \in \{Q, C\}$, and agent $i \in \{1, ..., N\}$, $\Pr(\theta_i \leq t \mid \mathbf{S} = \mathbf{s}; \mathbf{a})$ is twice differentiable wrt s_i and once differentiable wrt $\mathbf{s}_{-i}$ for every $(t, \mathbf{s}, \mathbf{a})$.*

Let $f_{\varepsilon+\theta^\perp}$ be the convolution of $f_{\theta^\perp}$ and f_ε, and similarly let $f_{\theta+\varepsilon^\perp}$be the convolution of f_θ and $f_{\varepsilon^\perp}$

Assumption 3. *For each $f \in \{f_{\varepsilon+\theta^\perp}, f_{\theta+\varepsilon^\perp}\}$, there exists a $\overline{\Delta} > 0$ and a function $J : \mathbb{R} \to \mathbb{R}_+$ such that*

$$\left(\frac{1}{\Delta}\frac{f(z-\Delta) - f(z)}{f(z)}\right)^2 \leq J(z)$$

for all $z \in \mathbb{R}$ and $\Delta \in (0, \overline{\Delta})$ and

$$\int J(z) f(z)\, dz < \infty.$$

Assumption 3 is a slight strengthening of the assumption that the Fisher information of S_i about $\overline{\theta}$ in the linked quality model and about $\overline{\varepsilon}$ in the linked circumstance model is finite. Roughly, it ensures that finite-difference approximations to the Fisher information are also finite and uniformly bounded as the approximation becomes more precise.[5]

[5] A sufficient condition for Assumption 3 is that the densities $f_{\varepsilon+\theta^\perp}$ and $f_{\theta+\varepsilon^\perp}$ don't vanish at the tails "much faster" than their derivatives: specifically, for each $f \in \{f_{\varepsilon+\theta^\perp}, f_{\theta+\varepsilon^\perp}\}$ there should exist a $K > 0$

Assumption 4. *The density functions $f_{\overline{\theta}}$, $f_{\theta^\perp}$, $f_{\overline{\varepsilon}}$, and $f_{\varepsilon^\perp}$ are strictly log-concave.*[6]

One basic property of strictly log-concave densities is that the convolution of two log-concave densities is also strictly log-concave. Thus an immediate corollary of Assumption 4 is that the densities f_θ and f_ε are strictly log-concave as well.

We impose this assumption to ensure that higher outcome realizations are good news about both the underlying types θ_i and shocks ε_i. In general, given three random variables X, Y, Z such that $X = Y + Z$ and Y and Z are independent, strict log-concavity of the density function of Z is both necessary and sufficient for the distribution of X to satisfy a strict monotone likelihood-ratio property in Y (Saumard and Wellner, 2014):

$$\frac{f_{X|Y}(x' \mid y')}{f_{X|Y}(x \mid y')} > \frac{f_{X|Y}(x' \mid y')}{f_{X|Y}(x \mid y)} \quad \text{if and only if} \quad x' > x, y' > y.$$

This monotone likelihood-ratio property is the canonical sufficient condition ensuring monotonicity of the conditional expectation of Y in the observed value of X (Milgrom, 1981). Assumption 4 guarantees that the appropriate monotone likelihood-ratio properties are satisfied in our model; see Appendix B.1 for details.

Finally, we assume the cost function is "sufficiently convex" so that effort choices satisfying an appropriate first-order condition are globally optimal. The assumption is jointly imposed on the cost function and the distribution of the outcome, since the required amount of convexity depends on how sensitive the posterior expectation is to the realization of individual outcomes.

Assumption 5. *There exists a $K \in \mathbb{R}$ such that $C''(x) > K$ for every $x \in \mathbb{R}_+$, population size N, and agent $i \in \{1, ..., N\}$, $\frac{\partial^2}{\partial s_i^2}\mathbb{E}[\theta_i \mid \mathbf{S} = \mathbf{s}; \mathbf{a}] \leq K$ for every $(\mathbf{s}, \mathbf{a})$.*

One important set of models satisfying these regularity conditions is the class of Gaussian models.[7]

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and $\overline{\Delta} > 0$ such that:

$$\max_{\varepsilon \in \mathbb{R}, \Delta \in [0, \overline{\Delta}]} \left| \frac{f'(\varepsilon - \Delta)}{f(\varepsilon)} \right| \leq K.$$

This sufficient condition is satisfied, for example, by the t-distribution and the logistic distribution. It is *not* satisfied by the normal distribution, although we show in Appendix E.1 using other methods that the normal distribution does satisfy Assumption 3. We have not been able to find any commonly-used distributions that fail Assumption 3.

[6] A function $g > 0$ is *strictly log-concave* if $\log g$ is strictly concave.

[7] The Gaussian versions of our signal structures represent special cases of the information environment considered in Bergemann, Bonatti and Gan (2019). In particular, the case of $\sigma_\varepsilon^2 = 0$ in Bergemann, Bonatti and Gan (2019) returns our linked quality model, while the case of $\sigma_\theta^2 = 0$ returns our linked circumstance model. The Gaussian version of our linked quality signal structure also corresponds to a symmetric version of the environment considered in Acemoglu et al. (2019).

Example. For each agent i,

$$\begin{pmatrix} \overline{\theta} \\ \theta_i^{\perp} \\ \overline{\varepsilon} \\ \varepsilon_i^{\perp} \end{pmatrix} \sim \mathcal{N}\left(\begin{pmatrix} \mu \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \sigma_{\overline{\theta}}^2 & 0 & 0 & 0 \\ 0 & \sigma_{\theta^{\perp}}^2 & 0 & 0 \\ 0 & 0 & \sigma_{\overline{\varepsilon}}^2 & 0 \\ 0 & 0 & 0 & \sigma_{\varepsilon^{\perp}}^2 \end{pmatrix} \right).$$

We verify in Appendix E.1 that Assumptions 1 through 4 are all met in this case, and Assumption 5 is satisfied by any strictly concave cost function.

3 Preliminary Results: Known Segment Size

We begin our analysis by studying a related model, in which the number of agents who opt-in, N, is exogenously given and commonly known. In this section we describe the derivation of equilibrium for this model. An accompanying formal analysis is provided in Appendix B.2.

3.1 Marginal Value of Effort

In equilibrium, agents choose effort such that the marginal impact of effort on the principal's forecast in the second period, which we will refer to as the *marginal value of effort*, equals its marginal cost. Here we define the marginal value of effort and its properties.

Let Ω denote the state space of realizations of types θ_i, noise shocks ε_i, and outcomes S_i for each agent $i = 1, ..., N$. Fix an equilibrium effort profile $(a_1^*, ..., a_N^*)$. The principal believes that each outcome is distributed $S_i = a_i^* + \theta_i + \varepsilon_i$, and any agent i who chooses the equilibrium effort level a_i^* believes the same. But if some agent i deviates to a non-equilibrium action $a_i = a_i^* + \Delta$, then he no longer shares the principal's prior over Ω. Specifically, agent i knows that his outcome is distributed $S_i = a_i^* + \Delta + \theta_i + \varepsilon_i$. This means that the agent's expected period-2 reward (i.e. the agent's expectation of the principal's forecast of his type) is an iterated expectation with respect to two different probability measures over Ω.

Formally, let $\mathbb{E}^{\Delta}$ denote expectations over Ω when agent i chooses effort level $a_i^* + \Delta$. For any profile of realized outcomes, the principal's expectation of agent i's type is

$$\mathbb{E}^0[\theta_i \mid S_1, \ldots, S_N].$$

If agent i exerts effort $a_i = a_i^* + \Delta$, then his expectation of the principal's forecast is

$$\mu_N(\Delta) \equiv \mathbb{E}^{\Delta}[\mathbb{E}^0[\theta_1 \mid S_1, \ldots, S_N]].$$

Note that the agent's expected reward when choosing the equilibrium effort level a_i^* is

$$\mu_N(0) = \mathbb{E}^0[\mathbb{E}^0[\theta_1 \mid S_1, \ldots, S_N]] = \mu \tag{2}$$

by the law of iterated expectations, reflecting the usual martingale property of posterior expectations. That is, in the absence of distortion away from the equilibrium effort a_i^*, the agent expects to receive μ in the second period.

When $\Delta \neq 0$, posterior expectations under the principal's beliefs are *not* a martingale from agent 1's perspective. In fact, as we show in Appendix B.2, $\mu_N(\Delta)$ is strictly increasing in Δ. Thus, increasing effort beyond the expected effort level always leads to a higher expected value of the principal's expectation. (This monotonicity is a consequence of the log-concavity property imposed in Assumption 4.)[8]

We define the marginal value of effort $MV(N)$ to be

$$MV(N) \equiv \mu_N'(0).$$

Our notation reflects the fact that $\mu_N(\Delta)$, thus also $MV(N)$, is independent of the levels of the equilibrium effort $a_1^*, ..., a_N^*$, due to the additive dependence of outcomes on effort. (We establish this property formally in Appendix B.2.)

These quantities can be expressed in closed form in the case of Gaussian unknowns.

Example. In the Gaussian model described in Section 2.5, an agent who exerts effort $a_i = a^* + \Delta$ expects the principal's forecast of his type to be

$$\mu_N(\Delta) = \mu + \beta(N) \cdot \Delta$$

for a function $\beta(N)$ that is independent of Δ. Linearity of $\mu_N(\Delta)$ in this example follows from an important property of Gaussian sampling that posterior means are linear in signal realizations. As a result, every extra unit of effort above the equilibrium level a^* increases the (expected) principal forecast of the agent's type by a constant amount $\beta(N)$. The marginal value of extra effort at the equilibrium effort level, $\mu_N'(0)$, is simply this constant slope; that is, $MV(N) = \beta(N)$. See Appendix E.2 for the closed-form expressions for $\beta(N)$ in each of the two models.

[8]This finding complements a result in Kartik, Lee and Suen (2019): If two agents update beliefs in response to common information about an unknown state, where the agents have different priors over the state but share a common perception of the signal structure, then the more optimistic agent expects the other's expectation of the state to increase with information. Here, we consider an agent and a principal who share the same prior over the agent types, but disagree over the signal structure.

Throughout, we use $MV_Q(N)$ and $MV_C(N)$ to denote the marginal value functions in the linked quality and circumstance models, respectively, dropping the subscript when a statement holds in both models. Note that $MV_Q(1) = MV_C(1)$; that is, with a single agent, the incentives for effort in the two models are identical.

3.2 Equilibrium Effort

The symmetry of our model ensures that all agents share the same marginal value and marginal cost of effort. There is therefore a unique effort level $a^*(N)$ satisfying each agent's equilibrium first-order condition

$$MV(N) = C'(a^*(N)) \tag{3}$$

equating the marginal value of effort $MV(N)$ with its equilibrium marginal cost $C'(a^*(N))$. This condition is both necessary and sufficient to ensure that each agent's optimal effort choice is indeed $a^*(N)$, when the principal expects all agents to exert effort $a^*(N)$.[9] The unique equilibrium of the exogenous-entry model then consists of choice of the effort level $a^*(N)$ by every agent. We write $a^*_Q(N)$ and $a^*_C(N)$ when we specifically mean the equilibrium action in (respectively) the linked quality model or the linked circumstance model.

The following key lemma characterizes the dependence of the marginal value function on the number of agents N:

Lemma 1. *(a) $MV_Q(N)$ is strictly decreasing in N and* $\lim_{N\to\infty} MV_Q(N) > 0$.

(b) $MV_C(N)$ is strictly increasing in N and $\lim_{N\to\infty} MV_C(N) < 1$.

That is, equilibrium effort is weakly decreasing in the number of agents in the linked quality model, and weakly increasing in the linked circumstance model. Observing that $a^*(N) = C'^{-1}(MV(N))$ with C'^{-1} strictly increasing, the same properties are inherited by the equilibrium actions $a^*(N)$:

Corollary 1. *(a) $a^*_Q(N)$ is nonincreasing in N and* $\lim_{N\to\infty} a^*_Q(N) > 0$.

(b) $a^*_C(N)$ is nondecreasing in N and $\lim_{N\to\infty} a^*_C(N) < 1$.

[9] The quantity $MV(N)$ captures the *local* value of distorting effort near the equilibrium value. Assumption 5 ensures that no deviation from equilibrium effort can yield higher returns, net of costs, than a local deviation, justifying the focus on local deviations from equilibrium effort.

Intuitively, the number of observations N influences how sensitive the principal's expectation of θ_i is to the realization of S_i. All else equal, the stronger the dependence of the forecast on i's outcome, the stronger the incentive to manipulate its distribution. In the linked circumstance model, other agents' data (which inform about the common part of the noise term $\overline{\varepsilon}$) complements agent i's outcome, improving its informativeness. Thus, the larger N is, the more weight the principal puts on i's outcome. This incentivizes effort. In the limit of large N, the principal learns $\overline{\varepsilon}$ and can de-bias the outcomes accordingly, so the incentives for agent i to exert effort are the same as in a single-agent model with $S_i = \theta_i + \varepsilon_i^{\perp}$.

By contrast, in the linked quality model, other agents' data (which inform about the common part of the type $\overline{\theta}$) substitutes for i's signal; thus, the larger N is, the less weight the principal puts on the realization of i's outcome. This de-incentivizes effort. In the limit as $N \to \infty$, the principal can extract $\overline{\theta}$ perfectly from the outcomes of other agents but retains uncertainty about $\theta_i^{\perp}$, so manipulation of S_i is still valuable. Specifically, the marginal value of effort is the same as in a single-agent model with $S_i = \overline{\theta} + \theta_i^{\perp} + \varepsilon_i$ and known $\overline{\theta}$.

Although this intuition is straightforward, we do not in general have access to the distribution of the principal's posterior expectation in closed form, so we cannot directly quantify the "strength" of the posterior expectation's dependence on the outcome S_i. Moreover, although it is straightforward to show that the sequence of functions $\mu_N(\Delta)$ converge *pointwise* to a limiting function $\mu_\infty(\Delta)$, the rates of this convergence may vary across Δ. Since we are primarily interested in the limiting marginal value $\lim_{N\to\infty} MV(N) = \lim_{N\to\infty} \mu'_N(0)$, we need the stronger property of *uniform* convergence of $\mu_N(\Delta)$ around $\Delta = 0$. We show that the expected impact of increasing effort by Δ, i.e. $\mu_N(\Delta) - \mu_N(0)$, can be bounded by an expression that shrinks (for Part (a)) or grows (for Part (b)) in N uniformly in Δ.[10] This establishes that the marginal value of deviating from equilibrium effort at finite N, $\mu'_N(0)$, indeed converges to the marginal value of effort in the limiting model, $\mu'_\infty(0)$. See Appendix C.1.2 for details.

[10] An implication of part (a) of Lemma 1 is that as $N \to \infty$, the agent's expectation of the principal's forecast converges to the agent's own expectation of his type; that is, μ. This implication has the flavor of the classic Blackwell and Dubins (1962) result on merging of opinions, which states that if two agents have different prior beliefs which are absolutely continuous with respect to one another, then given sufficient information, their posterior beliefs must converge. However, the Blackwell and Dubins (1962) result demonstrates almost-sure convergence, while we are interested in l_1-convergence under a shifted measure—that is, whether the agent's expectation of the principal's expectation converges to the agent's own expectation given sufficient data, where the agent and principal use different priors. Neither of these two notions of convergence directly imply the other.

4 Main Results

We now return to the main model, where the segment size N is endogenously determined in equilibrium by agents' opt-in decisions.

4.1 Equilibrium

In equilibrium, the principal correctly de-biases the observed outcomes. The equilibrium payoff in the second period is thus the prior mean μ (see (2)), and opt-in is (weakly) optimal if and only if the agent's equilibrium action a^* satisfies

$$R + \mu - C(a^*) \geq 0.$$

We impose the following assumption, which guarantees that agents would find it optimal to opt-in to a segment of size one. This restricts attention to settings in which a functioning market existed prior to identification of linkages across consumers.[11]

Assumption 6 (Individual Entry). *$R + \mu \geq C(a^*(1))$, where $a^*(1)$ is the equilibrium effort in the exogenous-entry game with a single agent (as defined in (3) with $N = 1$).*

Note that there is always a trivial no-entry equilibrium in which every agent chooses to opt-out, and the principal expects high effort from any agent who deviates to opting in. To refine away equilibria supported by potentially unreasonable beliefs, we require that if a principal observes deviation to opt-in by a single agent, he expects the action $a^*(1)$ that would have been chosen in the single-player exogenous-entry game.

Our main results characterize how the equilibrium implications of data sharing differ across the two models:

Theorem 1. *In the linked quality model, there is a unique symmetric equilibrium for all population sizes N. In this equilibrium, each agent opts-in and chooses effort level $a^*_Q(N) \in [0, a^*(1))$, where $a^*_Q(N)$ is strictly decreasing in N and $\lim_{N\to\infty} a^*_Q(N) > 0$.*

Theorem 2. *In the linked circumstance model, there exists an $N^* \in \mathbb{R}_+ \cup \{\infty\}$ such that:*

- *If $N \leq N^*$, there is a unique symmetric equilibrium in which each agent opts-in and chooses effort $a^*_C(N) > a^*(1)$, where $a^*_C(N)$ is strictly increasing in N.*

[11]The role of this assumption is primarily to simplify exposition. In the absence of this restriction, opt-in equilibria can fail to exist for small N.

- *If $N > N^*$, there is a unique symmetric equilibrium in which each agent opts-in with probability $p(N) \in (0,1)$ and chooses effort $a^{**} \in [a_C^*(N^*), a_C^*(N^*+1))$. The effort level a^{**} is independent of N, while the opt-in probability $p(N)$ is strictly decreasing in N.*

N^* is increasing in R, and is finite for R sufficiently small.

These equilibrium actions are depicted in Figure 1.

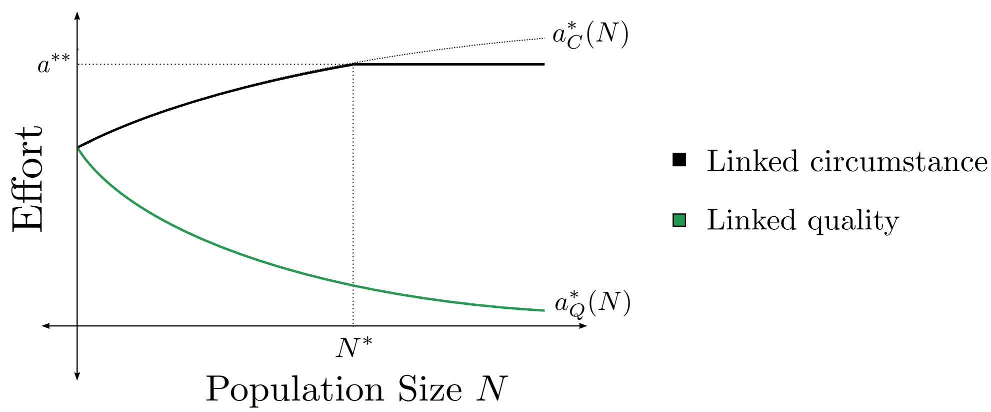

Figure 1: The relationship between population size and equilibrium effort

When the population size is small ($N < N^*$), all agents opt-in in both models, and the equilibrium effort levels $a_Q^*(N)$ and $a_C^*(N)$ are the same as in the previous section. Thus, the equilibrium effort levels inherit the properties described in Corollary 1.

As the population size grows large, opting-in becomes more attractive in the linked quality model, since equilibrium effort $a_Q^*(N)$ decreases in N. As a result, all agents participate no matter how large the population size. But in the linked circumstance model, effort $a_C^*(N)$ increases in N and so participation becomes less attractive as the population of entering agents grows. If N is large enough that the total costs of participation $C(a^*(N))$ exceed the expected reward $R + \mu$, then full participation cannot be an equilibrium. We let N^* denote the largest N for which $R + \mu \geq C(a^*(N))$. For any $N > N^*$, agents must mix over entry in equilibrium.[12]

Whenever $N > N^*$ and agents are connected by a linked circumstance, in equilibrium agents must enter at a rate $p(N) < 1$ and exert an effort level a^{**} which satisfy two conditions:

1. Agents are indifferent over entry:

$$R + \mu = C(a^{**}),$$

[12]If the opt-in reward R is large enough, it may be that $N^* = \infty$ and all agents enter no matter how large the population, as even the limiting effort level for very large populations is worth incurring for the large entry reward. The value N^* is finite whenever R is not too large.

2. The marginal value of distortion equals its marginal cost:

$$\mathbb{E}\left[MV(\widetilde{N}) \;\middle|\; \widetilde{N} \sim \text{Bin}(N, p(N))\right] = C'(a^{**})$$

The entry condition pins down the action level a^{**}, which is independent of the population size. The entry rate $p(N)$ is then pinned down by the requirement that the expected marginal value of effort must equal the marginal cost when agents who enter take action level a^{**}. Since the expected marginal value of effort rises with the number of entering agents, $p(N)$ must drop with N to equilibrate marginal values and costs. In general, this probability $p(N)$ is not the same as the probability $p^*(N)$ satisfying

$$MV\left(p^*(N) \cdot N\right) = C(a^{**}) \tag{4}$$

i.e. the opt-in probability such that equilibrium effort is a^{**} given *deterministic* entry of $p^*(N) \cdot N$ agents. We show in Appendix E.3 that $p(N) > p^*(N)$ for all N in the case of Gaussian unknowns, implying that uncertainty in the realized population size increases entry.

Theorems 1 and 2 reveal that the benefits to an organization from identifying consumer segments depends both on the nature of the linkage and on the number of agents in the segment. When agents within a segment have correlated quality—for example because they share a common lifestyle or other persistent attributes—then use of the linkage for prediction will lead to depressed effort by agents. An organization that values effort may therefore prefer to commit *not* to use big data analytics for forecasting outcomes based on such linkages. On the other hand, when agents experience shared circumstances (that affect outcomes but are not reflective of underlying quality), then use of the linkage will boost agent effort but potentially reduce participation. Whether the organization benefits from use of such linkages for prediction then depends on how it trades off between the goals of effort and participation. For consumers, data linkages regarding quality lead to higher equilibrium payoffs for all segment sizes, while data linkages regarding circumstance lead to (weakly) lower equilibrium payoffs for all segment sizes.

In practice, it may be unrealistic to expect that agents know the details of their segment, such as the precise nature of the correlation between types or shocks and how many other agents are present in their segment. However, as long as agents know whether their segment shares a linked quality versus a linked circumstance, the basic intuitions and all qualitative results go through. In particular, suppose that agents may be grouped into any of K "linked quality" segments, each of which corresponds to a different correlation structure across types; that is, $\overline{\theta} \sim F^k_{\overline{\theta}}$, $\theta^{\perp}_i \sim F^k_{\theta^{\perp}}$, and $\varepsilon_i \sim F^k_{\varepsilon}$ for segment $k = 1, ..., K$. All agents share a common belief about the probability that they are in each segment. (The case of K "linked circumstance" segments may be similarly defined.) At the same time, suppose that the number of agents N is a random variable, potentially dependent on the segment, with distribution $N \sim F^{k,\gamma}_N$, where γ is a scale factor such that for each segment k, $F^{k,\gamma}_N$ first-order stochastically dominates $F^{\gamma'}_N$ whenever $\gamma > \gamma'$. Then the following corollary holds:

Corollary. *In the model with uncertainty over segment and population size, equilibrium effort and participation rates exhibit the same comparative statics in γ as with respect to N in Theorems 1 and 2.*

Our main results therefore hinge only on the nature of the correlation between outcomes. (The threshold N^* at which participation rates begin to drop in the "linked circumstance" case would, however, depend on details of their beliefs about the segment.)

We now proceed to a more formal analysis of the welfare consequences of data linkages, focusing on a particular specification of social welfare that we will describe.

4.2 Welfare Implications

Consider any symmetric strategy profile (p, a) chosen by a population of N agents, where p is the opt-in probability and a is an action choice. We define total expected welfare from this strategy profile to be

$$\begin{aligned} W(p, a, N) &= \mathbb{E}\left[\sum_{i=1}^{N} x_i \left[S_i + \mathbb{E}\left(\theta_i \mid S_j, j \in \mathscr{I}_{\text{opt-in}}\right) - C(a)\right]\right] \\ &= pN \cdot (a + 2\mu - C(a)). \end{aligned}$$

where $x_i \in \{I, O\}$ is agent i's opt-in/out decision.

Following Holmstrom (1982), this welfare measure includes the following contributing factors: the agent's outcome S_i, the agent's expected second-period value $\mathbb{E}[\theta_i \mid S_j, j \in \mathscr{I}_{\text{opt-in}}]$, and the agent's effort cost $C(a_i)$. Different from Holmstrom (1982), we include these factors *only* for participating agents. The reward R does not enter into the welfare function, as we interpret it as a transfer between the principal and agent.

From Theorems 1 and 2, equilibrium in the linked quality model yields welfare

$$W_Q(N) \equiv W(1, a^*_Q(N), N)$$

while equilibrium in the linked circumstance model yields welfare

$$W_C(N) \equiv \begin{cases} W(1, a_C^*(N), N) & \text{if } N \leq N^* \\ W(p(N), a^{**}, N) & \text{otherwise} \end{cases}$$

Below we compare these quantities against two benchmarks:

Comparison with First-Best. Social welfare is maximized when all agents opt-in and choose the effort level a_{FB} satisfying $C'(a_{FB}) = 1$. Thus define

$$W_{FB}(N) \equiv W(1, a_{FB}, N)$$

We first show that equilibrium actions are below the first-best action in both models:

Proposition 1. *For every population size N, equilibrium effort is inefficiently low in both models:*

$$a^*(N) < a_{FB}.$$

As N increases:

- *Effort in the linked circumstance model $a_C^*(N)$ becomes more efficient but is bounded below the efficient level:* $\lim_{N\to\infty} a_C^*(N) < a_{FB}$
- *Effort in the linked quality model $a_Q^*(N)$ becomes less efficient.*

Recall that the equilibrium action a^* satisfies $C'(a^*) = MV(N)$ while the first-best action a_{FB} satisfies $C'(a_{FB}) = 1$. The proposition is proved by demonstrating that $MV(N) < 1$ in both models for all N. Intuitively, some effort is always dissipated, since the realization of the outcome is noisy, so the principal's forecast of θ_i moves less than 1-to-1 with the outcome. This result generalizes a classic result from Holmstrom (1982), which (in our notation) demonstrated that $a^*(1) < a_{FB}$ in the case of Gaussian random variables. An immediate corollary is that welfare is below the first-best in both models:

Corollary 2. *For every population size N, $W_Q(N), W_C(N) < W_{FB}$.*

Thus, in neither case are data linkages a way of recovering first-best welfare.

No Data Linkages. We next compare the equilibrium welfare with a "no data linkages" benchmark in which the principal does not understand the linkages across agents, and uses only agent i's outcome S_i to predict their type θ_i. That is, the principal's forecast is $\mathbb{E}(\theta_i \mid S_i)$.

In equilibrium, each agent opts-in (by Assumption 6), and chooses effort level

$$a_{NDL} \equiv a^*(1) \tag{5}$$

i.e. the action that would be taken (in either model) for a population of size 1. Welfare in the no-sharing welfare benchmark is then given by:

$$W_{NDL}(N) \equiv W(1, a_{NDL}, N)$$

Further taking into account the probability of entry, the subsequent corollary provides a comparison of welfare:

Proposition 2. *Let N^* be as defined in Theorem 2. There exists a threshold $2 \leq \overline{N} \leq \infty$ such that*

$$W_Q(N) < W_{NDL}(N) < W_C(N)$$

for all $1 < N < \overline{N}$ while

$$W_C(N), W_Q(N) < W_{NDL}(N)$$

for all $N > \overline{N}$. If $N^* \geq 2$, then $\overline{N} > 2$, and if $N^* < \infty$, then $\overline{N} < \infty$.

Equilibrium welfare is always strictly below the first best. Additionally, for all populations with $N \geq 2$ agents, the no-data linkages welfare benchmark $W_{NDL}(N)$ exceeds equilibrium welfare under the linked quality model. Thus, if players share a linkage over quality, aggregation of consumer data *always* leads to a reduction in social welfare. This follows directly from Proposition 1, since there is full entry in the no-data linkages benchmark as well as in the linked quality equilibrium, so the welfare comparison is completely determined by the relative sizes of the equilibrium actions $a_{NDL}(N) > a_Q^*(N)$.

In contrast, if the correlation across outcomes is described via the linked circumstance model, then the comparison depends on the population size N. In small populations, there is full-entry, so again the action comparison completely determines welfare. Since $a_{NDL}(N) < a_C^*(N)$, data linkages leads to an improvement in social welfare. In large populations, depressed entry dominates and results in lower social welfare, despite increased effort levels from participating agents.

These results suggest that regulations should take into account in what way the data is used to aid predictions. In particular, we should restrict use of big data to identify linkages over quality while encouraging use of big data to identify *linkages over circumstance* that are shared by small populations.

5 Comparative Statics

In the main text, we focused on a comparative static in the number of agents, which governed in part the size of the informational externality: the larger the segment, the more informative other agents' outcomes are about any given agent. The strength of this informational externality is also modulated by how "strong" the linkage is—i.e. how correlated the types or noise shocks are within the consumer segment—as well as by how precise the measured outcomes are—i.e. how much idiosyncratic noise is present in the outcomes. We now consider comparative statics in these parameters.

5.1 Strength of Linkage Across Agents

Consumer segments differ in the strength of the identified relationship. For example, consumers in the segment "Young Single Parents" may have more correlated repayment rates than consumers in the segment "Outdoor Hunting/Shooting." How does the strength of this linkage affect equilibrium outcomes?

We suppose that in the linked quality model, each agent i's type θ_i can be decomposed as

$$\theta_i = \sum_{j=1}^{M} \theta_i^j$$

where θ_i^1 through θ_i^M are independent and strictly log-concave. (Different components θ_i^m and $\theta_i^{m'}$ need not be identically distributed, though a single component θ_i^m should be identically distributed across agents to preserve symmetry.) Variables θ_i^1 through θ_i^m are common while θ_i^{m+1} through θ_i^M are independent across agents, so $\overline{\theta} = \theta_i^1 + \cdots + \theta_i^m$ while $\theta_i^{\perp} = \theta_i^{m+1} + \ldots \theta_i^M$. Likewise, in the linked circumstance model, suppose that ε_i can be decomposed as a sum of independent, strictly log-concave variables

$$\varepsilon_i = \sum_{j=1}^{M} \varepsilon_i^j,$$

where ε_i^1 through ε_i^m are common while ε_i^{m+1} through ε_i^M are idiosyncratic. The parameter m indexes the strength of the linkage, where a higher m corresponds to a stronger linkage.

Claim 1. *Equilibrium effort in the linked quality model is decreasing in the strength of the linkage, while equilibrum effort in the linked circumstance model is increasing in the strength of the linkage.*

In the linked quality model, stronger correlation across types means that more of the realized outcome S_i can be attributed to $\overline{\theta}$ rather than the idiosyncratic part $\theta_i^{\perp}$. Thus, other agents' outcomes are more informative about the common component $\overline{\theta}$. Moreover, when the linkage is stronger, S_i is less informative about $\theta_i = \overline{\theta} + \theta_i^{\perp}$ once $\overline{\theta}$ is understood. These effects work in the same direction, so the incentive to exert effort diminishes as the correlation across types strengthens.

In the linked circumstance model, stronger correlation across noise shocks means that other agents' outcomes are more informative about the common noise component $\overline{\varepsilon}$. This allows the principal to more accurately de-bias agent i's outcome. Additionally, agent i's outcome S_i is more informative about θ_i conditioning on removal of $\overline{\varepsilon}$. These two effects again work in the same direction, so that stronger correlation in the noise shock leads to higher incentive to exert effort.

Thus, the qualitative differences between linkages over quality and circumstance found in the main text are exaggerated when linkages are stronger: Strong linkages over quality result in an even greater reduction in effort, and strong linkages over circumstance result in an even greater increase in effort (compared to the no-linkage benchmark). To the extent that we expect identified consumer segments to become more accurate and more relevant to the organization's prediction problems as predictive modeling improves, strong linkages may increasingly be the relevant case.

5.2 Improved Monitoring

We next suppose that the principal gains access to an improved technology allowing for further de-noising of each agent's outcome, and characterize how this changes incentives for effort and equilibrium rates of entry.

We use a similar decomposition to the previous section. In the linked quality model, suppose that each player i's idiosyncratic noise term can be decomposed

$$\varepsilon_i = \sum_{j=1}^{M} \varepsilon_i^j,$$

where $M \geq 2$ and ε_i^1 through ε_i^M are strictly log-concave and mutually independent. This decomposition of the error term ε_i can be interpreted as separately accounting for the contribution of a number of different fluctuating environmental variables which affect the outcome of service provision. For instance, a driver's insurance claims may depend on the number of miles he has driven in the last coverage period, the roads on which this driving took place, and the weather and traffic conditions at the time of driving. We use the same formulation for the linked circumstance model, but decompose the idiosyncratic error term $\varepsilon_i^{\perp}$ instead.

Under an improved monitoring technology, the principal gains the ability to directly observe several of the idiosyncratic error shocks confounding each agent's type. Without loss of generality, suppose that the principal directly observes the first m variables, where improved monitoring corresponds to higher m.[13]

The following claim summarizes how an improvement in monitoring technology impacts equilibrium outcomes.

Claim 2. *(a) In the linked circumstance model, improved monitoring increases equilibrium effort and decreases entry. Welfare rises when the segment size N is sufficiently small, and decreases otherwise.*

(b) In the linked quality model, the impact of improved monitoring on equilibrium effort and entry is ambiguous. In the Gaussian model, improved monitoring decreases equilibrium effort, does not impact entry, and reduces welfare for any segment size N.

Consider first the linked circumstance model. In this model improved monitoring reduces the effective confound obscuring both $\overline{\varepsilon}$ and θ_i. This allows for improved inference of $\overline{\varepsilon}$ from other agents' outcomes, and also directly makes S_i a more informative signal about θ_i. Equilibrium effort therefore at least weakly increases, and strictly increases whenever all agents enter. Above the full-entry threshold, equilibrium entry rates must drop to maintain the zero-profit equilibium effort level given the higher marginal value of effort at every segment size. Below the full-entry threshold, raising effort levels moves effort closer to the first-best level, improving welfare. By contrast, above the full-entry threshold, effort is unchanged but entry rates drop, reducing welfare.

Now consider the linked quality model. In this model, improved monitoring has two competing effects. First, it makes S_i a more precise signal of $\theta_i = \overline{\theta} + \overline{\theta}_i^{\perp}$, and thus increases the marginal value of effort *holding beliefs about $\overline{\theta}$ fixed*. On the other hand, other agents' outcomes are more informative about $\overline{\theta}$, increasing the substitution effect. The net effect of an improvement in monitoring depends on a balancing of these two forces. In the Gaussian

[13] One might alternatively attempt to model improved monitoring via a scaling of the idiosyncratic error term, so that the principal observes $S_i' = \theta_i + \varepsilon_i/M$ for some scale factor M. However, S_i' is not guaranteed to be a more informative signal of θ_i than S_i (in the Blackwell order) unless ε_i can be decomposed into the sum of a random variable with distribution ε_i/M and another independent random variable. Our construction bypasses this difficulty by imposing the desired decomposability directly. Note that for Gaussian models ε_i is always decomposable in this way, so the two approaches are equivalent for that class of models.

model, the net effect is that the expectation of θ_i is less sensitive to S_i as monitoring improves, so equilibrium effort drops. Since all agents enter for any segment size in equilibrium, entry rates do not change with lower effort, and the effect of better monitoring is a drop in welfare for all segment sizes.

6 Conclusion

As organizations and governments move towards collecting and using large datasets of consumer transactions and behavior for decision-making, the question of whether and how to regulate data sharing has emerged as an important policy question.

Recent regulations, such as the European Union's General Data Protection Regulation (GDPR), have focused on protecting consumers' privacy and improving transparency regarding what kind of data is being collected. These policies are motivated by a goal of preserving basic rights, e.g. a right to privacy or a right to know what others know about you. Alternatively, regulations can be designed based on externalities or market consequences of uses of big data. In the present paper, we ask what effect consumer segments identified by big data have on consumer incentives for socially valuable effort.

We find that the behavioral consequences depends on whether consumers in a segment have correlated types or circumstances. When quality is correlated, then consumers exert lower effort in equilibrium than they would have without the identified linkage. This raises consumer payoffs, but leads to lower overall social welfare. In contrast, if consumers within a segment have correlated shocks, so that the segment helps the organization to de-noise observed outcomes, then effort is increased in equilibrium. This implies also that rates of participation with the organization may decrease, since outcomes in which all agents participate and exert high effort are not sustainable. The implications for social welfare depend on a comparison of how much effort is increased versus how much participation rates drop.

These results suggest that regulations should take into account not just whether individual data is informative about other consumers, but *how* it is informative. That relationship is crucial to how data linkages reshape incentives. In practice, whether data is used to predict a common type, or used for de-biasing other observations, is likely to differ across different domains, and may have as much to do with the underlying correlation structure of the data as it does with the algorithms used to aggregate that data. We do not explicitly model those algorithms here, although that is an interesting question for subsequent work.

Appendix

A Consumer Segments Provided by Data Brokers

In this Appendix we reproduce a list of examples of actual consumer segmentations produced by data brokers, as reported in Federal Trade Commission (2014) and Senate Committee on Commerce, Science, and Transportation (2013). We roughly categorize each segment according to whether consumers in the segment are more likely to exhibit a linked quality or a linked circumstance. In practice, the classification of a segment may depend on the time frame over which prediction takes place; for example, whether parents with children in college are linked by quality versus circumstance depends on the time interval of interest over which the principal must forecast consumer behavior.

Table 1: Examples of Consumer Segments

Linked Quality	**Linked Circumstance**
Outdoor/Hunting & Shooting	Sending a Kid to College
Santa Fe/Native American Lifestyle	Expectant Parents
Media Channel Usage - Daytime TV	Buying a Home
Bible Lifestyle	Getting Married
New Age/Organic Lifestyle	Dieters
Plus-size Apparel	Families with Newborns
Biker/Hell's Angels	Hard Times
Leans Left	New Mover/Renter/Owner
Exercise - Sporty Living	Death in the Family
Working-class Mom	
Thrifty Elders	
Health & Wellness Interest	
Very Spartan	
Small Town Shallow Pockets	
Established Elite	
Frugal Families	
McMansions & Minivans	

B Preliminary Results

B.1 Regularity of Posterior Distributions

In this appendix we establish important technical properties of the posterior distribution and mean of each agent's type, conditioning on the set of outcomes. These results provide important technical tools underpinning the results of the main text.

For the results of this section, fix a segment size N, and assume that all agents opt in. (All results extend immediately to any set of agents $I \subset \{1, ..., N'\}$ of size N entering from a segment of size $N' > N$.) Let G_i^M be the marginal distribution of agent i's outcome in model $M \in \{Q, C\}$, with $M = P$ the linked quality model and $M = T$ the linked circumstance model. We will write g_i^M for the density function associated with G_i^M, which is guaranteed to exist given that f_θ, f_η, and f_ε exist.

Definition B.1. *A family of density functions $\{f(\cdot \mid y)\}_{y \in Y}$ on $\mathbb{R}$ for some $Y \subset \mathbb{R}$ satisfies the* smooth monotone likelihood ratio property (SMLRP) *in y if:*

- *$f(\cdot \mid y)$ is a strictly positive, C^1 function for every y,*
- *There exists an $M < \infty$ such that $\left|\frac{\partial}{\partial x} f(x \mid y)\right| \leq M$ for every x and y,*
- *The function*
$$\ell(x; y, y') \equiv \frac{f(x \mid y)}{f(x \mid y')}$$
satisfies $\frac{\partial \ell}{\partial x}(x; y, y') > 0$ for every x and $y > y'$.

Lemma B.1. *Let X and Y be two independent random variables with density functions f_X and f_Y which are each C^1, strictly positive, strictly log-concave functions, and which each have bounded first derivative. Let $Z = k + X + Y$ for a constant k. Then the conditional densities $f_{Z|X}(z \mid x)$ and $f_{Z|Y}(z \mid y)$ satisfy the SMLRP in x and y, respectively.*

Proof. First take $k = 0$. We prove the result for $f_{Z|X}$, with the result for $f_{Z|Y}$ following symmetrically. Note that $f_{Z|X}(z \mid x) = f_Y(z - x)$. Then the fact that f_Y is strictly positive, C^1, and has bounded first derivative means that $f_{Z|X}$ satisfies the first two conditions of SMLRP. As for the likelihood ratio condition,

$$\frac{\partial}{\partial z} \frac{f_{Z|X}(z \mid x)}{f_{Z|X}(z \mid x')} = \frac{\partial}{\partial z} \frac{f_Y(z - x)}{f_Y(z - x')}$$

may be equivalently written

$$\frac{\partial}{\partial z} \frac{f_Y(z - x)}{f_Y(z - x')} = \frac{f_Y(z - x)}{f_Y(z - x')} \left(\frac{\partial}{\partial z} \log f_Y(z - x) - \frac{\partial}{\partial z} \log f_Y(z - x') \right).$$

Since f_Y is strictly log-concave, $\frac{\partial}{\partial z}\log f_Y(z-x) > \frac{\partial}{\partial z}\log f_Y(z-x')$ whenever $z-x < z-x'$, i.e. whenever $x' > x$. So the likelihood ratio condition is satisfied as well.

Now suppose $k \neq 0$. Then the result applied to the random variable $X+Y$ establishes that $f_{X+Y|X}(z \mid x)$ and $f_{X+Y|Y}(z \mid y)$ satisfy the SMLRP in x and y, respectively. As $f_{Z|X}(z \mid x) = f_{X+Y|X}(z-k \mid x)$ and $f_{Z|Y}(z \mid y) = f_{X+Y|Y}(z-k \mid y)$, and since since each of the conditions of the SMLRP condition are invariant to shifts in the first argument, these densities satisfy the SMLRP as well. □

Lemma B.2. *Let X and Y be two random variables whose family of conditional densities $\{f(x \mid y)\}$ exists and satisfies the SMLRP in y. Then for every y, the function $H(x) \equiv \Pr(Y \leq y \mid X = x)$ is differentiable for all x and satisfies $H'(x) < 0$.*

Proof. Let G be the distribution function for Y. By Bayes' rule,

$$H(x) = \frac{\int_{-\infty}^{y} f(x \mid y')\, dG(y')}{\int_{-\infty}^{\infty} f(x \mid y'')\, dG(y'')}.$$

Let $\widehat{H}(x) \equiv H(x)^{-1} - 1$. Then $H'(x)$ exists and satisfies $H'(x) < 0$ iff $\widehat{H}'(x)$ exists and satisfies $\widehat{H}'(x) > 0$. Note that $\widehat{H}(x)$ may be written

$$\widehat{H}(x) = \frac{\int_{y}^{\infty} f(x \mid y')\, dG(y')}{\int_{-\infty}^{y} f(x \mid y'')\, dG(y'')}.$$

Because $\left|\frac{\partial}{\partial x} f(x \mid y)\right|$ is uniformly bounded above, the Leibniz integral rule ensures that this expression is differentiable wrt x with derivative

$$\widehat{H}'(x) = \frac{\int_{y}^{\infty} \frac{\partial}{\partial x} f(x \mid y')\, dG(y')}{\int_{-\infty}^{y} f(x \mid y'')\, dG(y'')} - \frac{\left(\int_{y}^{\infty} f(x \mid y')\, dG(y')\right)\left(\int_{-\infty}^{y} \frac{\partial}{\partial x} f(x \mid y'')\, dG(y'')\right)}{\left(\int_{-\infty}^{y} f(x \mid y'')\, dG(y'')\right)^2}.$$

With some rearrangement, this may be equivalently written

$$\begin{aligned}\widehat{H}'(x) = &\left(\int_{-\infty}^{y} f(x \mid y'')\, dG(y'')\right)^{-2} \\ &\times \int_{y}^{\infty} dG(y') \int_{-\infty}^{y} dG(y'') \left(f(x \mid y'')\frac{\partial}{\partial x} f(x \mid y') - f(x \mid y')\frac{\partial}{\partial x} f(x \mid y'')\right).\end{aligned}$$

The integrand may be rewritten

$$\begin{aligned}&f(x \mid y'')\frac{\partial}{\partial x} f(x \mid y') - f(x \mid y')\frac{\partial}{\partial x} f(x \mid y'') \\ &= f(x \mid y'')^2 \left(\frac{\frac{\partial}{\partial x} f(x \mid y')}{f(x \mid y'')} - \frac{f(x \mid y')\frac{\partial}{\partial x} f(x \mid y'')}{f(x \mid y'')^2}\right) \\ &= f(x \mid y'')^2 \frac{\partial}{\partial x}\ell(x; y', y'').\end{aligned}$$

Now, as $y' > y > y''$ on the interior of the domain of integration, $\frac{\partial}{\partial x}\ell(x; y', y'') > 0$ everywhere and so $\widehat{H}'(x) > 0$, as desired. $\square$

Lemma B.3. *For every agent $i \in \{1, ..., N\}$, $\frac{\partial}{\partial S_i} F^Q_{\overline{\theta}}(\overline{\theta} \mid \mathbf{S}; \mathbf{a})$ exists and is strictly negative for all $(\overline{\theta}, \mathbf{S}, \mathbf{a})$, and $\frac{\partial}{\partial S_i} F^C_{\overline{\varepsilon}}(\overline{\varepsilon} \mid \mathbf{S}; \mathbf{a})$ exists and is strictly negative for all $(\overline{\varepsilon}, \mathbf{S}, \mathbf{a})$.*

Proof. For convenience we suppress the dependence of distributions on $\mathbf{a}$ in ths proof. This lemma follows from Lemma B.2 provided that $g^Q_i(S_i \mid \overline{\theta}, \mathbf{S}_{-i})$ and $g^C_i(S_i \mid \overline{\varepsilon}, \mathbf{S}_{-i})$ satisfy SMLRP wrt $\overline{\theta}$ and $\overline{\varepsilon}$ respectively. But conditional on $\overline{\theta}$, S_i is independent of $\mathbf{S}_{-i}$ in the linked quality model, and similarly conditional on $\overline{\varepsilon}$, S_i is independent of $\mathbf{S}_{-i}$ in the linked circumstance model. So it suffices to establish that $g^Q_i(S_i \mid \overline{\theta})$ satisfies SMLRP wrt $\overline{\theta}$ while $g^C_i(S_i \mid \overline{\varepsilon})$ satisfies SMLRP wrt $\overline{\varepsilon}$.

Recall that in the linked quality model, $S_i = \overline{\theta} + \theta^{\perp}_i + \varepsilon_i$, where by assumption $\overline{\theta}$, $\theta^{\perp}_i$ and ε_i all have C^1, strictly positive, strictly log-concave density functions with bounded derivatives. Then the same is true for the density function of the sum $\theta^{\perp}_i + \varepsilon_i$, which is just the convolution of the density functions for $\theta^{\perp}_i$ and ε_i. In that case Lemma B.1 implies that $g^Q_i(S_i \mid \overline{\theta})$ satisfies SMLRP wrt $\overline{\theta}$, as desired. An analogous line of reasoning establishes the result for the linked circumstance model. $\square$

Lemma B.4. $\mathbb{E}[\theta_1 \mid \mathbf{S}; \mathbf{a}]$ *is differentiable with respect to S_1 for every $(\theta_1, \mathbf{S}, \mathbf{a})$, and*

$$0 < \frac{\partial}{\partial S_1}\mathbb{E}[\theta_1 \mid \mathbf{S}; \mathbf{a}] < 1.$$

Proof. For convenience, we suppress the dependence of distributions on $\mathbf{a}$ in this proof. We establish the result for the linked quality model, with the result for the linked circumstance model following by nearly identical work.

Fix a vector of signal realizations $\mathbf{S}_{-1}$. First note that $g^Q_1(S_1 \mid \theta_1, \mathbf{S}_{-1}) = g^Q_1(S_1 \mid \theta_1)$, and S_1 is the sum of a constant plus the independent random variables θ_1 and ε_1, each of which has a strictly positive, C^1 density function with bounded derivative. Thus by Lemma B.1 $g^Q_1(S_1 \mid \theta_1, \mathbf{S}_{-1})$ satisfies SMLRP wrt θ_1. Meanwhile Conditional on this realization, S_1 can be written

$$S_1 = a_1 + \overline{\theta}' + \theta^{\perp}_1 + \varepsilon_1,$$

where $\overline{\theta}' = \mathbb{E}[\overline{\theta} \mid \mathbf{S}_{-1}]$ has the density function $\widetilde{f}_{\overline{\theta}}$ defined by $\widetilde{f}_{\overline{\theta}}(t) \equiv f^Q_{\overline{\theta}}(\overline{\theta} = t \mid \mathbf{S}_{-1})$, and $\overline{\theta}', \theta^{\perp}_1$, and ε_1 are mutually independent. We first show that $\widetilde{f}_{\overline{\theta}}$ is a C^1, strictly positive, strictly log-concave function with bounded derivative. By Bayes' rule,

$$\widetilde{f}_{\overline{\theta}}(t) = \frac{f_{\overline{\theta}}(t)\prod_{i>1} g^Q_i(S_i \mid \overline{\theta} = t)}{g^Q(\mathbf{S}_{-1})} = \frac{f_{\overline{\theta}}(t)\prod_{i>1} f_{\varepsilon+\theta^{\perp}}(S_i - t - a_i)}{g^Q(\mathbf{S}_{-1})},$$

where $f_{\varepsilon+\theta^\perp}$ is the convolution of $f_{\theta^\perp}$ and f_ε. Since $f_{\theta^\perp}$ and f_ε are both strictly positive, strictly log-concave functions with bounded derivative, so is $f_{\varepsilon+\theta^\perp}$. It follows immediately that $\widetilde{f}_{\overline{\theta}}$ is a strictly positive, C^1 function with bounded derivative. Further, taking logarithms yields

$$\log \widetilde{f}_{\overline{\theta}}(t) = \log f_{\overline{\theta}}(t) - \log g^Q(\mathbf{S}_{-1}) + \sum_{i>1} \log f_{\varepsilon+\theta^\perp}(S_i - t - a_i).$$

Hence $\log \widetilde{f}_{\overline{\theta}}$ is a sum of constant and strictly concave functions, meaning it is strictly concave. Thus $\widetilde{f}_{\overline{\theta}}$ is strictly log-concave. This means that conditional on $\mathbf{S}_{-1}$, S_1 is the sum of a constant plus the independent random variables ε_1 and $\theta' + \theta_1^\perp$, each of which has a strictly positive, C^1 density function with bounded derivative. So by Lemma B.1 $g_1^Q(S_1 \mid \varepsilon_1, \mathbf{S}_{-1})$ satisfies SMLRP wrt ε_1.

In light of the work of the previous paragraph, Lemma B.2 ensures that $\frac{\partial}{\partial S_1} F^Q_{\theta_1}(\theta_1 \mid \mathbf{S}) < 0$ and $\frac{\partial}{\partial S_1} F^Q_{\varepsilon_1}(\varepsilon_1 \mid \mathbf{S}) < 0$ everywhere. Now, by definition,

$$\mathbb{E}[\theta_1 \mid \mathbf{S}] = \int_{-\infty}^{\infty} \theta_1 \, dF^Q_{\theta_1}(\theta_1 \mid \mathbf{S}).$$

Using integration by parts, this may be equivalently written

$$\mathbb{E}[\theta_1 \mid \mathbf{S}] = \int_0^{\infty} (1 - F^Q_{\theta_1}(\theta_1 \mid \mathbf{S}))\, d\theta_1 - \int_{-\infty}^{0} F^Q_{\theta_1}(\theta_1 \mid \mathbf{S})\, d\theta_1.$$

Suppose enough regularity to exchange derivatives and integrals. Then

$$\frac{\partial}{\partial S_1}\mathbb{E}[\theta_1 \mid \mathbf{S}] = -\int_{-\infty}^{\infty} \frac{\partial}{\partial S_1} F^M_{\theta_1}(\theta_1 \mid \mathbf{S})\, d\theta_1 > 0.$$

By very similar work,

$$\frac{\partial}{\partial S_1}\mathbb{E}[\varepsilon_1 \mid \mathbf{S}] > 0.$$

Finally, recall that

$$S_1 = a_1 + \theta_1 + \varepsilon_1,$$

so that

$$S_1 = a_1 + \mathbb{E}[\theta_1 \mid \mathbf{S}] + \mathbf{E}[\varepsilon_1 \mid \mathbf{S}].$$

Differentiating both sides yields

$$1 = \frac{\partial}{\partial S_1}\mathbb{E}[\theta_1 \mid \mathbf{S}] + \frac{\partial}{\partial S_1}\mathbb{E}[\varepsilon_1 \mid \mathbf{S}].$$

Since each term on the rhs is strictly positive, each much also be strictly less than 1. $\square$

Lemma B.5. $\mathbb{E}[\theta_1 \mid \mathbf{S};\mathbf{a}]$ *is differentiable with respect to* S_N *for every* $(\theta_1, \mathbf{S}, \mathbf{a})$*, and*

$$\frac{\partial}{\partial S_N}\mathbb{E}[\theta_1 \mid \mathbf{S};\mathbf{a}] > 0$$

in the linked quality model, while

$$\frac{\partial}{\partial S_N}\mathbb{E}[\theta_1 \mid \mathbf{S};\mathbf{a}] < 0$$

in the linked circumstance model.

Proof. For convenience, we suppress the dependence of distributions on $\mathbf{a}$ in this proof. By definition,

$$\mathbb{E}[\theta_1 \mid \mathbf{S}] = \int_{-\infty}^{\infty} \theta_1 \, dF^M_{\theta_1}(\theta_1 \mid \mathbf{S}).$$

Using integration by parts, this may be equivalently written

$$\mathbb{E}[\theta_1 \mid \mathbf{S}] = \int_0^{\infty} (1 - F^M_{\theta_1}(\theta_1 \mid \mathbf{S}))\, d\theta_1 - \int_{-\infty}^{0} F^M_{\theta_1}(\theta_1 \mid \mathbf{S})\, d\theta_1.$$

Suppose that $F^M_{\theta_1}(\theta_1 \mid \mathbf{S})$ is differentiable wrt S_N, with enough regularity that the dominated convergence theorem applies. Then

$$\frac{\partial}{\partial S_N}\mathbb{E}[\theta_1 \mid \mathbf{S}] = -\int_{-\infty}^{\infty} \frac{\partial}{\partial S_N} F^M_{\theta_1}(\theta_1 \mid \mathbf{S})\, d\theta_1.$$

It is therefore sufficient to sign the derivative in the interior of the integral.

Consider first the linked quality model. Then

$$F^Q_{\theta_1}(\theta_1 \mid \mathbf{S}) = \int_{-\infty}^{\infty} F^Q_{\theta_1}(\theta_1 \mid \overline{\theta})\, dF^Q_{\overline{\theta}}(\overline{\theta} \mid \mathbf{S}).$$

Using integration by parts, this can be equivalently written

$$F^Q_{\theta_1}(\theta_1 \mid \mathbf{S}) = \lim_{\overline{\theta} \to -\infty} F^Q_{\theta_1}(\theta_1 \mid \overline{\theta}) + \int_{-\infty}^{\infty} (1 - F^Q_{\overline{\theta}}(\overline{\theta} \mid \mathbf{S})) \frac{\partial}{\partial \overline{\theta}} F^Q_{\theta_1}(\theta_1 \mid \overline{\theta})\, d\overline{\theta}.$$

Now, $F^Q_{\theta_1}(\theta_1 \mid \overline{\theta}) = F^Q_{\theta_1^{\perp}}(\theta_1 - \overline{\theta})$, so because $\theta_1^{\perp}$ has a bounded, strictly positive density function, $\frac{\partial}{\partial \overline{\theta}} F^Q_{\theta_1}(\theta_1 \mid \overline{\theta})$ is strictly negative and uniformly bounded for all $(\theta_1, \overline{\theta})$. Meanwhile by Lemma B.3 $\frac{\partial}{\partial S_N} F^Q_{\overline{\theta}}(\overline{\theta} \mid \mathbf{S})$ exists and is strictly negative for all $(\overline{\theta}, \mathbf{S})$. Then assuming enough regularity to exchange derivatives and integrals,

$$\frac{\partial}{\partial S_N} F^Q_{\theta_1}(\theta_1 \mid \mathbf{S}) = -\int_{-\infty}^{\infty} \frac{\partial}{\partial S_N} F^Q_{\overline{\theta}}(\overline{\theta} \mid \mathbf{S}) \frac{\partial}{\partial \overline{\theta}} F^Q_{\theta_1}(\theta_1 \mid \overline{\theta})\, d\overline{\theta} < 0$$

everywhere, implying

$$\frac{\partial}{\partial S_N}\mathbb{E}[\theta_1 \mid \mathbf{S}] > 0$$

everywhere.

Now consider the linked circumstance model. In this case

$$F^C_{\theta_1}(\theta_1 \mid \mathbf{S}) = \int_{-\infty}^{\infty} F^C_{\theta_1}(\theta_1 \mid \mathbf{S}, \overline{\varepsilon})\, dF^C_{\overline{\varepsilon}}(\overline{\varepsilon} \mid \mathbf{S}).$$

Conditional on $\overline{\varepsilon}$, θ_1 depends on $\mathbf{S}$ only through S_1, so this can be written

$$F^C_{\theta_1}(\theta_1 \mid \mathbf{S}) = \int_{-\infty}^{\infty} F^C_{\theta_1}(\theta_1 \mid S_1, \overline{\varepsilon})\, dF^C_{\overline{\varepsilon}}(\overline{\varepsilon} \mid \mathbf{S}).$$

Using integration by parts, this is equivalently

$$F^C_{\theta_1}(\theta_1 \mid \mathbf{S}) = \lim_{\overline{\varepsilon}\to-\infty} F^C_{\theta_1}(\theta_1 \mid S_1, \overline{\varepsilon}) + \int_{-\infty}^{\infty} (1 - F^C_{\overline{\varepsilon}}(\overline{\varepsilon} \mid \mathbf{S}))\frac{\partial}{\partial \overline{\varepsilon}} F^C_{\theta_1}(\theta_1 \mid S_1, \overline{\varepsilon})\, d\overline{\varepsilon}.$$

Now, $F^C_{\theta_1}(\theta_1 \mid S_1 = s_1, \overline{\varepsilon} = z) = F^C_{\theta_1}(\theta_1 \mid \widetilde{S}_1 = s_1 - z)$, where $\widetilde{S}_1 \equiv \theta_1 + \varepsilon_1^{\perp}$. By Lemma B.1 the density of $\widetilde{S}_1$ conditional on θ_1 satisfies SMLRP wrt θ_1, so that by Lemma B.2 $\frac{\partial}{\partial \widetilde{S}_1} F^C_{\theta_1}(\theta_1 \mid \widetilde{S}_1) < 0$ everywhere. Hence $\frac{\partial}{\partial \overline{\varepsilon}} F^C_{\theta_1}(\theta_1 \mid S_1, \overline{\varepsilon}) > 0$ everywhere. Meanwhile by Lemma B.3 $\frac{\partial}{\partial S_N} F^C_{\overline{\varepsilon}}(\overline{\varepsilon} \mid \mathbf{S})$ exists and is strictly negative for all $(\overline{\varepsilon}, \mathbf{S})$. Then assuming enough regularity to exchange derivatives and integrals,

$$\frac{\partial}{\partial S_N} F^C_{\theta_1}(\theta_1 \mid \mathbf{S}) = -\int_{-\infty}^{\infty} \frac{\partial}{\partial S_N} F^C_{\overline{\varepsilon}}(\overline{\varepsilon} \mid \mathbf{S}) \frac{\partial}{\partial \overline{\varepsilon}} F^C_{\theta_1}(\theta_1 \mid S_1, \overline{\varepsilon})\, d\overline{\varepsilon} > 0$$

everywhere, implying

$$\frac{\partial}{\partial S_N}\mathbb{E}[\theta_1 \mid \mathbf{S}] < 0$$

everywhere. □

Lemma B.6. *For each model $M \in \{Q, C\}$, $\frac{\partial}{\partial S_1} G^M_N(S_N \mid \mathbf{S}_{-N}; \mathbf{a})$ exists and is strictly negative for all $(\mathbf{S}, \mathbf{a})$.*

Proof. For convenience, we suppress the dependence of distributions on $\mathbf{a}$ in this proof. We prove the result for the linked quality model, with the result for the linked circumstance model following by nearly identical work. In the linked quality model

$$G^Q_N(S_N \mid \mathbf{S}_{-N}) = \int_{-\infty}^{\infty} G^Q_N(S_N \mid \overline{\theta}, \mathbf{S}_{-N})\, dF^Q_{\overline{\theta}}(\overline{\theta} \mid \mathbf{S}_{-N}).$$

Conditional on $\overline{\theta}$, the distribution of S_N is independent of $\mathbf{S}_{-N}$, and further $G_N^Q(S_N \mid \overline{\theta}) = F_{\theta_N^\perp+\varepsilon_N}(S_N - \overline{\theta})$, so this expression may be written

$$G_N^Q(S_N \mid \mathbf{S}_{-N}) = \int_{-\infty}^{\infty} F^Q_{\theta_N^\perp+\varepsilon_N}(S_N - \overline{\theta})\, dF^Q_{\overline{\theta}}(\overline{\theta} \mid \mathbf{S}_{-N}).$$

Using integration by parts, this may be equivalently written

$$G_N^Q(S_N \mid \mathbf{S}_{-N}) = 1 - \int_{-\infty}^{\infty} (1 - F^Q_{\overline{\theta}}(\overline{\theta} \mid \mathbf{S}_{-N})) f^Q_{\theta_N^\perp+\varepsilon_N}(S_N - \overline{\theta})\, d\overline{\theta}.$$

Now, Lemma B.3 ensures that $\frac{\partial}{\partial S_1} F^Q_{\overline{\theta}}(\overline{\theta} \mid \mathbf{S}_{-N}))$ exists and is strictly negative everywhere. (The proof of the lemma does not depend on the number of signals or which signal is differentiated with respect to.) Then assuming enough regularity to exchange derivatives and integrals,

$$\frac{\partial}{\partial S_1} G_N^Q(S_N \mid \mathbf{S}_{-N}) = \int_{-\infty}^{\infty} \frac{\partial}{\partial S_1} F^Q_{\overline{\theta}}(\overline{\theta} \mid \mathbf{S}_{-N})) f^Q_{\theta_N^\perp+\varepsilon_N}(S_N - \overline{\theta})\, d\overline{\theta} < 0.$$

□

B.2 Sufficiency of the First-Order Approach

In this appendix we formally establish that there exists a unique equilibrium to the exogenous-entry model, which is characterized by the first-order condition described in the body of the paper. Fix a population size N, and assume all agents in the segment enter in the first period. For every $\alpha \in \mathbb{R}_+^N$ and $\Delta \geq -\alpha_1$, define

$$\mu(\Delta; \alpha) \equiv \mathbb{E}[\mathbb{E}[\theta_1 \mid \mathbf{S}; \mathbf{a} = \alpha] \mid \mathbf{a} = (\alpha_1 + \Delta, \alpha_{-1})]$$

to be agent 1's expected second-period payoff from exerting effort $\alpha_1 + \Delta$ when the principal expects each agent $i \in \{1, ..., N\}$ to exert effort α_i.

Lemma B.7. *The value function $\mu(\Delta; \alpha)$ and its derivatives satisfy the following properties:*

(a) $\mu(\Delta; \alpha)$ is independent of α and is continuous and strictly increasing in Δ.

(b) $\mu'(\Delta; \alpha)$ exists, is continuous in Δ, and satisfies $0 < \mu'(\Delta; \alpha) < 1$ for every Δ.

(c) $D^+\mu'(\Delta; \alpha) \leq K$ for every Δ.[14]

[14] Given a function $f : \mathbb{R} \to \mathbb{R}$, the Dini derivative D^+ is a generalization of the derivative existing for arbitrary functions and defined by $D^+ f(x) = \limsup_{h \downarrow 0}(f(x+h) - f(x))/h$. When f is differentiable at a point x, $D^+ f(x) = f'(x)$.

Proof. Fix a model $M \in \{Q, C\}$. The quantity $\mu(\Delta;\alpha)$ can be written explicitly as

$$\mu(\Delta;\alpha) = \int dG^M(\mathbf{S} = \mathbf{s} \mid \mathbf{a} = (\alpha_1 + \Delta, \alpha_{-1}))\, \mathbb{E}[\theta \mid \mathbf{S} = \mathbf{s}; \mathbf{a} = \alpha].$$

Further,

$$\mathbb{E}[\theta \mid \mathbf{S} = \mathbf{s}; \mathbf{a} = \alpha] = \int \theta_1 \, dF^M_{\theta_1}(\theta_1 \mid \mathbf{S} = \mathbf{s}; \mathbf{a} = \alpha),$$

and by Bayes' rule

$$f^M_{\theta_1}(\theta_1 \mid \mathbf{S} = \mathbf{s}; \mathbf{a} = \alpha) = \frac{g^M(\mathbf{S} = \mathbf{s} \mid \theta_1; \mathbf{a} = \alpha) f_\theta(\theta)}{g^M(\mathbf{S} = \mathbf{s} \mid \mathbf{a} = \alpha)}.$$

Since effort affects the outcome as an additive shift, $g^M(\mathbf{S} = \mathbf{s} \mid \mathbf{a} = \alpha) = g^M(\mathbf{S} = \mathbf{s} - \alpha \mid \mathbf{a} = \mathbf{0})$ and $g^M(\mathbf{S} = \mathbf{s} \mid \theta_1; \mathbf{a} = \alpha) = g^M(\mathbf{S} = \mathbf{s} - \alpha \mid \theta_1; \mathbf{a} = \mathbf{0})$. So

$$\begin{aligned} f^M_{\theta_1}(\theta \mid \mathbf{S} = \mathbf{s}; \mathbf{a} = \alpha) &= \frac{g^M(\mathbf{S} = \mathbf{s} - \alpha \mid \theta; \mathbf{a} = \mathbf{0}) f_\theta(\theta)}{g^M(\mathbf{S} = \mathbf{s} - \alpha \mid \mathbf{a} = \mathbf{0})} \\ &= f^M_{\theta_1}(\theta_1 \mid \mathbf{S} = \mathbf{s} - \alpha; \alpha = \mathbf{0}). \end{aligned}$$

Thus

$$\mathbb{E}[\theta_1 \mid \mathbf{S} = \mathbf{s}; \mathbf{a} = \alpha] = \int \theta_1 \, dF^M_{\theta_1}(\theta_1 \mid \mathbf{S} = \mathbf{s} - \alpha; \mathbf{a} = \mathbf{0}) = \mathbb{E}[\theta_1 \mid \mathbf{S} = \mathbf{s} - \alpha; \mathbf{a} = \mathbf{0}].$$

Then $\mu(\Delta;\alpha)$ may be equivalently written

$$\mu(\Delta;\alpha) = \int dG^M(\mathbf{S} = \mathbf{s} - \alpha \mid \mathbf{a} = (\Delta, \mathbf{0}))\, \mathbb{E}[\theta_1 \mid \mathbf{S} = \mathbf{s} - \alpha; \mathbf{a} = \mathbf{0}].$$

Using the change of variables $\mathbf{s}' = \mathbf{s} - \alpha$ then reveals that $\mu(\Delta;\alpha) = \mu(\Delta;\mathbf{0})$, so μ is indeed independent of α.

Now fix Δ and $\Delta' < \Delta$. Since effort affects the outcome as an additive shift, $G^M(\mathbf{S} = \mathbf{s} \mid \mathbf{a} = (\alpha_1 + \Delta, \alpha_{-1})) = G^M(\mathbf{S} = (s_1 - (\Delta - \Delta'), \mathbf{s}_{-1}) \mid \mathbf{a} = (\alpha_1 + \Delta, \alpha_{-1}))$ for every s_1. Then defining a change of variables via $s_1' = s_1 - (\Delta - \Delta')$ and $\mathbf{s}'_{-i} = \mathbf{s}_{-i}$, the previous integral expression for $\mu(\Delta;\alpha)$ may be equivalently written

$$\mu(\Delta;\alpha) = \int dG^M(\mathbf{S} = \mathbf{s}' \mid \mathbf{a} = (\alpha_1 + \Delta', \alpha_{-1}))\, \mathbb{E}[\theta_1 \mid \mathbf{S} = (s_1' + (\Delta - \Delta'), \mathbf{s}'_{-1}); \mathbf{a} = \alpha].$$

Now by Lemma B.4, $0 < \frac{\partial}{\partial S_1}\mathbb{E}[\theta_1 \mid \mathbf{S}; \mathbf{a}] < 1$ everywhere. Hence by the Leibniz integral rule $\mu'(\Delta;\alpha)$ exists and

$$\mu'(\Delta;\alpha) = \int dG^M(\mathbf{S} = \mathbf{s}' \mid \mathbf{a} = \alpha)\, \frac{\partial}{\partial \Delta}\mathbb{E}[\theta_1 \mid \mathbf{S} = (s_1' + \Delta, \mathbf{s}'_{-1}); \mathbf{a} = \alpha],$$

and in particular $0 < \mu'(\Delta;\alpha) < 1$. An immediate corollary is that $\mu(\Delta;\alpha)$ is continuous and strictly increasing everywhere.

Meanwhile, by Assumption 5

$$\frac{\partial^2}{\partial\Delta^2}\mathbb{E}[\theta_1 \mid \mathbf{S} = (s_1' + \Delta, \mathbf{s}_{-1}'); \mathbf{a} = \alpha]$$

exists and is bounded in the interval $(-\infty, K]$ everywhere. As $\frac{\partial}{\partial\Delta}\mathbb{E}[\theta_1 \mid \mathbf{S} = (s_1'+\Delta, \mathbf{s}_{-1}'); \mathbf{a} = \alpha]$ is also continuous in Δ everywhere, which along with uniform boundedness implies that $\mu'(\Delta;\alpha)$ is continuous in Δ everywhere by the dominated convergence theorem. Further, for each $\delta > 0$ and $(\mathbf{s}, \mathbf{a}, \Delta)$, the mean value theorem implies that

$$\begin{aligned}
&\frac{1}{\delta}\left(\frac{\partial}{\partial\Delta}\mathbb{E}[\theta_1 \mid \mathbf{S} = (s_1' + \Delta + \delta, \mathbf{s}_{-1}'); \mathbf{a} = \alpha] - \frac{\partial}{\partial\Delta}\mathbb{E}[\theta_1 \mid \mathbf{S} = (s_1' + \Delta, \mathbf{s}_{-1}'); \mathbf{a} = \alpha]\right) \\
&= \frac{\partial^2}{\partial\Delta^2}\mathbb{E}[\theta_1 \mid \mathbf{S} = (s_1' + \Delta + \delta', \mathbf{s}_{-1}'); \mathbf{a} = \alpha] \le K
\end{aligned}$$

for some $\delta' \in [0, \delta]$. Reverse Fatou's lemma then implies that $D^+\mu'(\Delta;\alpha) \le K$. □

Lemma B.8. $\mu(\Delta;\alpha) - C(\alpha_1 + \Delta)$ *is a strictly concave function of* Δ *for any* α.

Proof. Fix an α, and define $\phi(\Delta) \equiv \mu(\Delta;\alpha) - C(\alpha_1 + \Delta)$. By Lemma B.7, ϕ' exists and is continuous everywhere. We establish the necessary and sufficient condition for strict concavity that ϕ' is strictly decreasing. We invoke the basic monotonicity theorem from analysis that any function f which is continuous and satisfies $D^+f \ge 0$ everywhere is nondecreasing everywhere. We apply this result to $-\mu'(\Delta;\alpha) + K\Delta$. Using basic properties of the Dini derivatives D^+ and D_+, we have $D^+(-\mu'(\Delta;\alpha)) = -D_+\mu'(\Delta;\alpha) \ge -D^+\mu'(\Delta;\alpha)$. Then since $K\Delta$ is differentiable and $D^+\mu'(\Delta;\alpha) \le K$ from Lemma B.7, we have $D^+(-\mu'(\Delta;\alpha)+K\Delta) = D^+(-\mu'(\Delta;\alpha)) + K \ge 0$. So $\mu'(\Delta;\alpha) - K\Delta$ is nonincreasing everywhere. So choose any Δ and $\Delta' > \Delta$. Then

$$\phi'(\Delta') = \mu'(\Delta';\alpha) - K\Delta' + K\Delta' - C'(\alpha_1 + \Delta') \le \mu'(\Delta;\alpha) + K(\Delta' - \Delta) - C'(\alpha_1 + \Delta').$$

But also by Assumption 5, $C''(\alpha_1 + \Delta'') > K$ for every $\Delta'' \in (\Delta, \Delta')$, so $C'(\alpha_1 + \Delta') > C'(\alpha_1 + \Delta) + K(\Delta' - \Delta)$. Thus

$$\phi'(\Delta') < \mu'(\Delta;\alpha) - C'(\alpha_1 + \Delta) = \phi'(\Delta),$$

as desired. □

Proposition B.1. *There exists a unique equilibrium action profile characterized by* $a_i = a_i^*(N)$ *for each player* i*, where* $a_i^*(N)$ *is the unique solution to*

$$\mu'(0; \mathbf{a}^*(N)) = C'(a^*(N)).$$

Proof. Lemma B.7 established that $\mu'(0; \mathbf{a}^*(N))$ is well-defined, independent of $a^*(N)$, and bounded in the interval $[0, 1]$. Then as C' is continuous, strictly increasing, and satisfies $C'(0) = 0$ and $C'(\infty) = \infty$, there exists a unique solution to the stated first-order condition. This solution constitutes an equilibrium so long as $\Delta = 0$ maximizes the objective function $\mu(\Delta; \mathbf{a}^*(N)) - C(a^*(N) + \Delta)$, which is guaranteed by the fact, established in Lemma B.8, that this function is strictly concave in Δ. □

Define $\mu_N(\Delta) \equiv \mu(\Delta; \mathbf{a}^*(N))$ and $MV(N) \equiv \mu_N'(0)$ for each N. When we wish to make the model clear, we will write $MV_M(N)$ for $M \in \{Q, C\}$. An immediate implication of Lemma B.7 is that $0 < MV(N) < 1$ for all N. We conclude this appendix by establishing that these bounds also hold strictly in the limit as $N \to \infty$.

Lemma B.9. $0 < \lim_{N\to\infty} MV(N) < 1$.

Proof. For convenience we suppress the dependence of distributions on $\mathbf{a}$ in this proof. We prove the result for the linked quality model, with the result for the linked circumstance model following by nearly identical work.

The proof of Lemma 1 establishes that $\lim_{N\to\infty} MV(N) = MV(\infty)$, where $MV(\infty)$ is the equilibrium marginal value of effort in a one-agent model where the common component $\overline{\theta}$ is observed by the principal. But conditional on $\overline{\theta}$, S_1 is the sum of a constant plus the two independent random variables $\theta_1^\perp$ and ε_1, each of which has a C^1, strictly positive density function with bounded derivative. Thus by Lemma B.1 $g_1^Q(S_1 \mid \varepsilon_1, \overline{\theta})$ satisfies SMLRP wrt ε_1. And meanwhile $g_1^Q(S_1 \mid \theta_1, \overline{\theta}) = g_1^Q(S_1 \mid \theta_1)$, and unconditionally S_1 is the sum of the two independent random variables θ_1 and ε_1, each of which has a C^1, strictly positive density function with bounded derivative. So $g_1^Q(S_1 \mid \theta_1, \overline{\theta})$ satisfies SMLRP wrt θ_1. Then by Lemma B.2, $\frac{\partial}{\partial S_1} F_{\theta_1}^Q(\theta_1 \mid S_1, \overline{\theta}) < 0$ and $\frac{\partial}{\partial S_1} F_{\varepsilon_1}^Q(\varepsilon_1 \mid S_1, \overline{\theta}) < 0$ everywhere.

Now, by definition

$$\mathbb{E}[\theta_1 \mid S_1, \overline{\theta}] = \int_{-\infty}^{\infty} \theta_1 \, dF_{\theta_1}^Q(\theta_1 \mid S_1, \overline{\theta}).$$

Using integration by parts, this may be equivalently written

$$\mathbb{E}[\theta_1 \mid S_1, \overline{\theta}] = \int_0^{\infty} (1 - F_{\theta_1}^Q(\theta_1 \mid S_1, \overline{\theta})) \, d\theta_1 - \int_{-\infty}^{0} F_{\theta_1}^Q(\theta_1 \mid S_1, \overline{\theta}) \, d\theta_1.$$

Suppose enough regularity to exchange derivatives and integrals. Then

$$\frac{\partial}{\partial S_1}\mathbb{E}[\theta_1 \mid S_1, \overline{\theta}] = -\int_{-\infty}^{\infty} \frac{\partial}{\partial S_1} F_{\theta_1}^Q(\theta_1 \mid S_1, \overline{\theta}) \, d\theta_1 > 0.$$

By very similar work,

$$\frac{\partial}{\partial S_1}\mathbb{E}[\varepsilon_1 \mid S_1, \overline{\theta}] > 0.$$

Finally, recall that

$$S_1 = a_1 + \theta_1 + \varepsilon_1,$$

so that

$$S_1 = a_1 + \mathbb{E}[\theta_1 \mid S_1, \overline{\theta}] + \mathbb{E}[\varepsilon_1 \mid S_1, \overline{\theta}].$$

Differentiating both sides yields

$$1 = \frac{\partial}{\partial S_1}\mathbb{E}[\theta_1 \mid S_1, \overline{\theta}] + \frac{\partial}{\partial S_1}\mathbb{E}[\varepsilon_1 \mid S_1, \overline{\theta}].$$

Since each term on the rhs is strictly positive, each much also be strictly less than 1. $\square$

C Proofs for Section 3 (Exogenous Entry)

C.1 Proof of Lemma 1

Throughout this proof, we will without loss of generality consider agent 1's problem. To compare results across segments of differing sizes, we will consider there to be a single underlying vector $\mathbf{S} = (S_1, S_2, ...)$ of outcomes for a countably infinite set of agents, with the N-agent model observing the outcomes of the first N agents. Notationally, we will write $\mathbf{S}_{i:j}$ to indicate the vector of outcomes of agent i through agent j.

C.1.1 Monotonicity in N

We first establish the monotonicity claims of the lemma. Fix a model $M \in \{Q, C\}$ and a segment size N. By definition, the expected value of distortion $\mu_N(\Delta)$ is

$$\begin{aligned}\mu_N(\Delta) = &\int dG^M_{1:N}(\mathbf{S}_{1:N} = \mathbf{s}_{1:N} \mid \mathbf{a}_{1:N} = (a^*(N) + \Delta, \mathbf{a}^*(N)_{2:N})) \\ &\times \left(\int \theta_1 \, dF^M_{\theta_1}(\theta_1 \mid \mathbf{S}_{1:N} = \mathbf{s}_{1:N}; \mathbf{a}_{1:N} = \mathbf{a}^*(N)_{1:N})\right).\end{aligned}$$

Our first task is to rewrite this expression conditioning on the vector of actions $\mathbf{a}^*(N+1)_{1:N}$, to facilitate comparison with the value of distortion $\mu_{N+1}(\Delta)$ with one more agent. The additive structure of the model implies that the distribution functions $G^M_{1:N}$ and $F^M_{\theta_1}$ satisfy the identities

$$G^M_{1:N}(\mathbf{S}_{1:N} = \mathbf{s}_{1:N} \mid \mathbf{a}_{1:N}) = G^M_{1:N}(\mathbf{S}_{1:N} = \mathbf{s}_{1:N} + \delta\mathbf{a}_{1:N} \mid \mathbf{a}_{1:N} + \delta\mathbf{a}_{1:N})$$

and

$$F_{\theta_1}^M(\theta_1 \mid \mathbf{S}_{1:N} = \mathbf{s}_{1:N}; \mathbf{a}_{1:N}) = F_{\theta_1}^M(\theta_1 \mid \mathbf{S}_{1:N} = \mathbf{s}_{1:N} + \delta\mathbf{a}_{1:N}; \mathbf{a}_{1:N} + \delta\mathbf{a}_{1:N})$$

for any outcome realization $\mathbf{s}_{1:N}$ and any action vectors $\mathbf{a}_{1:N}$ and $\delta\mathbf{a}_{1:N}$. So let $\delta\mathbf{a}^*_{1:N} \equiv (\mathbf{a}^*(N+1) - \mathbf{a}^*(N) - \Delta, \mathbf{a}^*(N+1)_{2:N} - \mathbf{a}^*(N)_{2:N})$. Then the previous representation of $\mu_N(\Delta)$ may be equivalently written

$$\begin{aligned}\mu_N(\Delta) = &\int dG_{1:N}^M(\mathbf{S}_{1:N} = \mathbf{s}_{1:N} + \delta\mathbf{a}^*_{1:N} \mid \mathbf{a}_{1:N} = \mathbf{a}^*(N+1)_{1:N}) \\ &\quad \times \left(\int \theta_1 \, dF_{\theta_1}^M(\theta_1 \mid \mathbf{S}_{1:N} = \mathbf{s}_{1:N} + \delta\mathbf{a}^*_{1:N} + (\Delta, \mathbf{0}_{2:N}); \mathbf{a}_{1:N} = \mathbf{a}^*(N+1)_{1:N})\right).\end{aligned}$$

Changing variables to the integrator $\mathbf{s}'_{1:N} = \mathbf{s}_{1:N} + \delta\mathbf{a}^*_{1:N}$, this becomes

$$\begin{aligned}\mu_N(\Delta) = &\int dG_{1:N}^M(\mathbf{S}_{1:N} = \mathbf{s}'_{1:N} \mid \mathbf{a}_{1:N} = \mathbf{a}^*(N+1)_{1:N}) \\ &\quad \times \left(\int \theta_1 \, dF_{\theta_1}^M(\theta_1 \mid \mathbf{S}_{1:N} = (s'_1 + \Delta, \mathbf{s}'_{2:N}); \mathbf{a}_{1:N} = \mathbf{a}^*(N+1)_{1:N})\right).\end{aligned}$$

The value of distortion $\mu_{N+1}(\Delta)$ for a segment of $N+1$ agents may be similarly written

$$\begin{aligned}\mu_{N+1}(\Delta) = &\int dG_{1:N+1}^M(\mathbf{S}_{1:N+1} = \mathbf{s}'_{1:N+1} \mid \mathbf{a}_{1:N+1} = \mathbf{a}^*(N+1)) \\ &\quad \times \left(\int \theta_1 \, dF_{\theta_1}^M(\theta_1 \mid \mathbf{S}_{1:N+1} = (s'_1 + \Delta, \mathbf{s}'_{2:N+1}); \mathbf{a}_{1:N+1} = \mathbf{a}^*(N+1))\right).\end{aligned}$$

To compare these expressions, we use the law of iterated expectations. In the N-agent model we have

$$\begin{aligned}&\int \theta_1 \, dF_{\theta_1}^M(\theta_1 \mid \mathbf{S}_{1:N} = (s'_1 + \Delta, \mathbf{s}'_{2:N}); \mathbf{a}_{1:N} = \mathbf{a}^*(N+1)_{1:N}) \\ &= \int dG_{N+1}^M(S_{N+1} = s'_{N+1} \mid \mathbf{S}_{1:N} = (s'_1 + \Delta, \mathbf{s}'_{2:N}); \mathbf{a}_{1:N+1} = \mathbf{a}^*(N+1)) \\ &\quad \times \left(\int \theta_1 \, dF_{\theta_1}^M(\theta_1 \mid \mathbf{S}_{1:N+1} = (s'_1 + \Delta, \mathbf{s}'_{2:N+1}); \mathbf{a}_{1:N+1} = \mathbf{a}^*(N+1))\right).\end{aligned}$$

So

$$\begin{aligned}\mu_N(\Delta) = &\int dG_{1:N}^M(\mathbf{S}_{1:N} = \mathbf{s}'_{1:N} \mid \mathbf{a}_{1:N} = \mathbf{a}^*(N+1)_{1:N}) \\ &\quad \int dG_{N+1}^M(S_{N+1} = s'_{N+1} \mid \mathbf{S}_{1:N} = (s'_1 + \Delta, \mathbf{s}'_{2:N}); \mathbf{a}_{1:N+1} = \mathbf{a}^*(N+1)) \\ &\qquad \times \left(\int \theta \, dF_{\theta}^M(\theta \mid \mathbf{S}_{1:N+1} = (s'_1 + \Delta, \mathbf{s}'_{2:N+1}); \mathbf{a}_{1:N+1} = \mathbf{a}^*(N+1))\right).\end{aligned}$$

Meanwhile in the $N+1$-agent model we have

$$\mu_{N+1}(\Delta) = \int dG^M_{1:N+1}(\mathbf{S}_{1:N} = \mathbf{s}'_{1:N} \mid \mathbf{a}_{1:N} = \mathbf{a}^*(N+1)_{1:N}) \int dG^M_{N+1}(S_{N+1} = s'_{N+1} \mid \mathbf{S}_{1:N} = \mathbf{s}'_{1:N}; \mathbf{a}_{1:N+1} = \mathbf{a}^*(N+1)) \times \left(\int \theta_1 \, dF^M_{\theta_1}(\theta_1 \mid \mathbf{S}_{1:N+1} = (s'_1 + \Delta, \mathbf{s}'_{2:N+1}); \mathbf{a}_{1:N+1} = \mathbf{a}^*(N+1)) \right).$$

So define a function ψ by

$$\psi(\delta_1, \delta_2, \mathbf{s}_{1:N}) \equiv \int dG^M_{N+1}(S_{N+1} = s_{N+1} \mid \mathbf{S}_{1:N} = (s_1 + \delta_1, \mathbf{s}_{2:N}); \mathbf{a}_{1:N+1} = \mathbf{a}^*(N+1)) \times \left(\int \theta_1 \, dF^M_{\theta_1}(\theta_1 \mid \mathbf{S}_{1:N+1} = (s_1 + \delta_2, \mathbf{s}_{2:N+1}); \mathbf{a}_{1:N+1} = \mathbf{a}^*(N+1)) \right).$$

Then

$$\mu_N(\Delta) = \int dG^M_{1:N}(\mathbf{S}_{1:N} = \mathbf{s}'_{1:N} \mid \mathbf{a}_{1:N} = \mathbf{a}^*(N+1)_{1:N}) \, \psi(\Delta, \Delta, \mathbf{s}'_{1:N})$$

while

$$\mu_{N+1}(\Delta) = \int dG^M_{1:N}(\mathbf{S}_{1:N} = \mathbf{s}'_{1:N} \mid \mathbf{a}_{1:N} = \mathbf{a}^*(N+1)_{1:N}) \, \psi(0, \Delta, \mathbf{s}'_{1:N}).$$

Then as $\mu_N(0) = \mu_{N+1} = \mu$,

$$\begin{aligned} &MV(N) - MV(N+1) \\ = &\lim_{\Delta \to 0} \frac{1}{\Delta} (\mu_N(\Delta) - \mu_{N+1}(\Delta)) \\ = &\lim_{\Delta \to 0} \int dG^M_{1:N}(\mathbf{S}_{1:N} = \mathbf{s}'_{1:N} \mid \mathbf{a}_{1:N} = \mathbf{a}^*(N+1)_{1:N}) \, \frac{1}{\Delta} \left(\psi(\Delta, \Delta, \mathbf{s}'_{1:N}) - \psi(0, \Delta, \mathbf{s}'_{1:N}) \right). \end{aligned}$$

Now, let

$$H(\delta; \mathbf{s}_{1:N+1}) \equiv \int \theta_1 \, dF^M_{\theta_1}(\theta_1 \mid \mathbf{S}_{1:N+1} = (s_1 + \delta, \mathbf{s}_{2:N+1}); \mathbf{a}_{1:N+1} = \mathbf{a}^*(N+1))$$

and

$$j(\delta; \mathbf{s}_{1:N+1}) \equiv g^M_{N+1}(S_{N+1} = s'_{N+1} \mid \mathbf{S}_{1:N} = (s'_1 + \delta, \mathbf{s}'_{2:N}); \mathbf{a}_{1:N+1} = \mathbf{a}^*(N+1)).$$

Then

$$\begin{aligned} &\psi(\Delta, \Delta, \mathbf{s}'_{1:N}) - \psi(0, \Delta, \mathbf{s}'_{1:N}) \\ = &\int ds'_{N+1} \, (j(\Delta; \mathbf{s}'_{1:N+1}) - j(0; \mathbf{s}'_{1:N+1})) H(\Delta; \mathbf{s}'_{1:N+1}) \\ = &\int ds'_{N+1} \, (j(\Delta; \mathbf{s}'_{1:N+1}) - j(0; \mathbf{s}'_{1:N+1}))(H(\Delta; \mathbf{s}'_{1:N+1}) - H(0; \mathbf{s}'_{1:N+1})) \\ &+ \int ds'_{N+1} \, (j(\Delta; \mathbf{s}'_{1:N+1}) - j(0; \mathbf{s}'_{1:N+1})) H(0; \mathbf{s}'_{1:N+1}). \end{aligned}$$

Now, Lemma B.4 ensures that $H'(\delta;\mathbf{s}_{1:N+1})$ exists and is uniformly bounded in $(0,1)$ for all δ and $\mathbf{s}_{1:N+1}$. The mean value theorem then implies the bound

$$|H(\Delta;\mathbf{s}_{1:N+1}) - H(0;\mathbf{s}_{1:N+1})| \leq \Delta$$

for all $\mathbf{s}_{1:N+1}$. Further suppose that $j'(\delta;\mathbf{s}_{1:N+1})$ exists and is bounded in $[-M,M]$ for all δ and $\mathbf{s}_{1:N+1}$. The mean value theorem then implies the bound

$$|j(\Delta;\mathbf{s}_{1:N+1}) - j(0;\mathbf{s}_{1:N+1})| \leq M\Delta$$

for all $\mathbf{s}_{1:N+1}$. The previous bounds together with the triangle inequality then yield

$$\frac{1}{\Delta}|\psi(\Delta,\Delta,\mathbf{s}_{1:N}) - \psi(0,\Delta,\mathbf{s}_{1:N})| \leq M\Delta + M$$

for all $\mathbf{s}_{1:N}$ and Δ. Thus by the dominated convergence theorem

$$\begin{aligned}&MV(N) - MV(N+1)\\ &= \int dG^M_{1:N}(\mathbf{S}_{1:N} = \mathbf{s}'_{1:N} \mid \mathbf{a}_{1:N} = \mathbf{a}^*(N+1)_{1:N}) \lim_{\Delta\to 0}\frac{1}{\Delta}(\psi(\Delta,\Delta,\mathbf{s}'_{1:N}) - \psi(0,\Delta,\mathbf{s}'_{1:N})),\end{aligned}$$

supposing this inner limit exists. Further, the dominated convergence theorem implies that

$$\lim_{\Delta\to 0}\frac{1}{\Delta}\int ds'_{N+1}\,(j(\Delta;\mathbf{s}'_{1:N+1}) - j(0;\mathbf{s}'_{1:N+1}))(H(\Delta;\mathbf{s}'_{1:N+1}) - H(0;\mathbf{s}'_{1:N+1}) = 0$$

and

$$\lim_{\Delta\to 0}\frac{1}{\Delta}\int ds'_{N+1}\,(j(\Delta;\mathbf{s}'_{1:N+1}) - j(0;\mathbf{s}'_{1:N+1}))H(0;\mathbf{s}'_{1:N+1}) = \int ds'_{N+1}\,j'(0;\mathbf{s}'_{1:N+1})H(0;\mathbf{s}'_{1:N+1}),$$

so the limit

$$\lim_{\Delta\to 0}\frac{1}{\Delta}(\psi(\Delta,\Delta,\mathbf{s}'_{1:N}) - \psi(0,\Delta,\mathbf{s}'_{1:N}))$$

exists. It will be most convenient to write

$$\lim_{\Delta\to 0}\frac{1}{\Delta}(\psi(\Delta,\Delta,\mathbf{s}'_{1:N}) - \psi(0,\Delta,\mathbf{s}'_{1:N})) = \left.\frac{d}{d\Delta}\right|_{\Delta=0}\int ds'_{N+1}\,j(\Delta;\mathbf{s}'_{1:N+1})H(0;\mathbf{s}'_{1:N+1})$$

without exchanging the derivative and integral, so that $MV(N) - MV(N+1)$ may be expressed as

$$\begin{aligned}&MV(N) - MV(N+1)\\ &= \int dG^M_{1:N}(\mathbf{S}_{1:N} = \mathbf{s}'_{1:N} \mid \mathbf{a}_{1:N} = \mathbf{a}^*(N+1)_{1:N}) \left.\frac{d}{d\Delta}\right|_{\Delta=0}\int ds'_{N+1}\,j(\Delta;\mathbf{s}'_{1:N+1})H(0;\mathbf{s}'_{1:N+1}).\end{aligned}$$

To sign the difference between $MV(N)$ and $MV(N+1)$ it is therefore enough to sign the quantity

$$\frac{d}{d\Delta}\bigg|_{\Delta=0} \int ds_{N+1}\, j(\Delta; \mathbf{s}_{1:N+1}) H(0; \mathbf{s}_{1:N+1})$$

for fixed $\mathbf{s}_{1:N}$.

Now fix $\mathbf{s}'_{1:N}$, and let s^*_{N+1} be the unique solution to $H(0; \mathbf{s}'_{1:N}, s_{N+1}) = 0$. Also let

$$J(\delta; \mathbf{s}_{1:N}) \equiv G^M_{N+1}(S_{N+1} = s_{N+1} \mid \mathbf{S}_{1:N} = (s_1 + \delta, \mathbf{s}_{2:N}); \mathbf{a}_{1:N+1} = \mathbf{a}^*(N+1))$$

and

$$h(\mathbf{s}_{1:N+1}) \equiv \frac{\partial}{\partial s_{N+1}} \int \theta_1 \, dF^M_{\theta_1}(\theta_1 \mid \mathbf{S}_{1:N+1} = \mathbf{s}_{1:N+1}; \mathbf{a}_{1:N+1} = \mathbf{a}^*(N+1)).$$

Then integration by parts implies that

$$\begin{aligned}
&\int ds'_{N+1}\, j(\Delta; \mathbf{s}'_{1:N+1}) H(0; \mathbf{s}'_{1:N+1}) \\
&= \int_{s^*_{N+1}}^{\infty} ds'_{N+1}\, (1 - J(\Delta; \mathbf{s}'_{1:N+1})) h(\mathbf{s}'_{1:N+1}) \\
&\quad - \int_{-\infty}^{s^*_{N+1}} ds'_{N+1}\, J(\Delta; \mathbf{s}'_{1:N+1}) h(\mathbf{s}'_{1:N+1}).
\end{aligned}$$

Assuming enough regularity to exchange derivatives and limits,

$$\frac{d}{d\Delta}\bigg|_{\Delta=0} \int ds_{N+1}\, j(\Delta; \mathbf{s}_{1:N+1}) H(0; \mathbf{s}_{1:N+1}) = -\int_{-\infty}^{\infty} ds'_{N+1}\, J'(0; \mathbf{s}'_{1:N+1}) h(\mathbf{s}'_{1:N+1}).$$

Finally, Lemma B.6 ensures that $J'(0; \mathbf{s}'_{1:N+1}) < 0$ everywhere in both models, while Lemma B.5 ensures that $h(\mathbf{s}'_{1:N+1}) > 0$ everywhere in the linked quality model and $h(\mathbf{s}'_{1:N+1}) < 0$ everywhere in the linked circumstance model. So $MV(N) - MV(N+1) > 0$ in the linked quality model, while $MV(N) - MV(N+1) < 0$ in the linked circumstance model, as desired.

C.1.2 The $N \to \infty$ limit

Consider a limiting model in which the principal observes a countably infinite vector of outcomes $\mathbf{S} = (S_1, S_2, ...)$. By the law of large numbers, in the linked quality model this means that the principal perfectly infers $\overline{\theta}$, while in the linked circumstance model the principal perfectly infers $\overline{\varepsilon}$. Define $\mu(\Delta; \alpha)$ analogously to the finite-population case. In each model,reasoning very similar to the proof of Lemma B.7 implies that $\mu'(0, \alpha)$ exists, is independent of α, and lies in $[0, 1]$. So there exists a unique, finite $a^*(\infty)$ satisfying $\mu'(0; \mathbf{a}^*(\infty)) = C'(a^*(\infty))$. Define $\mu_\infty(\Delta) \equiv \mu(\Delta; \mathbf{a}^*(\infty))$ and $MV(\infty) \equiv \mu'_\infty(0)$ in each model. Lemma B.9

establishes that $0 < MV(\infty) < 1$. We will show that $\lim_{N\to\infty} MV(N) = MV(\infty)$. Lemma B.9 establishes that this result implies $0 < \lim_{N\to\infty} MV(N) < 1$.

To prove the result, we will need the ability to change measure between the distribution of outcomes at the equilibrium action profile, and one in which a single agent, without loss agent 1, deviates to a different action. For each model, define a reference probability space $(\Omega, \mathcal{F}, \mathcal{P}^{\mathbf{a}})$, containing all relevant random variables for arbitrary segment sizes. For the linked quality model this space supports the latent types $\overline{\theta}, \theta_1^{\perp}, \theta_2^{\perp}, ...$ and shocks $\varepsilon_1, \varepsilon_2, ...$ as well as the outcomes $S_1, S_2, ...$ Similarly, in the linked circumstance model the space supports the latent types $\theta_1, \theta_2, ...$, shocks $\overline{\varepsilon}, \varepsilon_1^{\perp}, \varepsilon_2^{\perp}, ...$, and outcomes $S_1, S_2, ...$ In each model the probability measure $\mathcal{P}^{\mathbf{a}}$ depends on the vector of agent actions $\mathbf{a} = (a_1, a_2, ...)$, as the distributions of the outcomes depend on the actions.

We will use $\mathcal{F}^{\infty}$ to denote the σ-algebra generated by the full vector of outcomes $S_1, S_2, ...$ Note that by the LLN all latent types may be taken to be measurable wrt $\mathcal{F}^{\infty}$. Finally, for each segment size N, we will let $\mathcal{P}^{*N}$ denote the restriction of the measure $\mathcal{P}^{\mathbf{a}^*(N)}$ to $(\Omega, \mathcal{F}^{\infty})$, and similarly let $\mathcal{P}^{\Delta,N}$ denote the restriction of the measure $\mathcal{P}^{(a^*(N)+\Delta, \mathbf{a}^*(N))}$ to $(\Omega, \mathcal{F}^{\infty})$. These measures represent the distributions over outcomes induced when all agents take actions $a^*(N)$ and when agent 1 deviates to action $a^*(N) + \Delta$, respectively.

Lemma C.1. *The Radon-Nikodym derivative for the change of measure from $(\Omega, \mathcal{F}^{\infty}, \mathcal{P}^{*N})$ to $(\Omega, \mathcal{F}^{\infty}, \mathcal{P}^{\Delta,N})$ is*

$$\frac{d\mathcal{P}^{\Delta,N}}{d\mathcal{P}^{*N}} = \frac{g_1^Q(S_1 \mid \overline{\theta}; a_1 = a^*(N) + \Delta)}{g_1^Q(S_1 \mid \overline{\theta}; a_1 = a^*(N))}$$

in the linked quality model and

$$\frac{d\mathcal{P}^{\Delta,N}}{d\mathcal{P}^{*N}} = \frac{g_1^C(S_1 \mid \overline{\varepsilon}; a_1 = a^*(N) + \Delta)}{g_1^C(S_1 \mid \overline{\varepsilon}; a_1 = a^*(N))}$$

in the linked circumstance model.

Proof. For convenience we suppress the dependence of distributions on all actions other than a_1 in this proof. We derive the derivative for the linked quality model, with the expression for the linked circumstance model following from nearly identical work. Fix any $\mathcal{F}^{\infty}$-measurable random variable X. Then there exists a measurable function $x : \mathbb{R}^{\infty} \to \mathbb{R}$ such that $X = x(\mathbf{S})$ a.s. Thus

$$\begin{aligned}
&\mathbb{E}[X \mid a_1 = a^*(N) + \Delta] \\
&= \int dF_{\overline{\theta}}(\overline{\theta})\, dG_1^Q(S_1 \mid \overline{\theta}; a_1 = a^*(N) + \Delta)\, dG_{-1}^Q(\mathbf{S}_{-1} \mid \overline{\theta}, S_1; a_1 = a^*(N) + \Delta) \\
&\qquad \times x(\mathbf{S}).
\end{aligned}$$

As $\mathbf{S}_{-1}$ is independent of S_1 conditional on $\overline{\theta}$ in the linked quality model, $G^Q_{-1}(\mathbf{S}_{-1} \mid \overline{\theta}, S_1; a_1 = a^*(N) + \Delta) = G^Q_{-1}(\mathbf{S}_{-1} \mid \overline{\theta})$. So this expression may be equivalently written

$$\begin{aligned}
&\mathbb{E}[X \mid a_1 = a^*(N) + \Delta] \\
&= \int dF_{\overline{\theta}}(\overline{\theta})\, dG^Q_1(S_1 \mid \overline{\theta}; a_1 = a^*(N) + \Delta)\, dG^Q_{-1}(\mathbf{S}_{-1} \mid \overline{\theta})\, x(\mathbf{S}) \\
&= \int dF_{\overline{\theta}}(\overline{\theta})\, dG^Q_1(S_1 \mid \overline{\theta}; a_1 = a^*(N))\, dG^Q_{-1}(\mathbf{S}_{-1} \mid \overline{\theta}) \\
&\qquad \times \frac{g^Q_1(S_1 \mid \overline{\theta}; a_1 = a^*(N) + \Delta)}{g^Q_1(S_1 \mid \overline{\theta}; a_1 = a^*(N))} x(\mathbf{S}) \\
&= \mathbb{E}\left[\frac{g^Q_1(S_1 \mid \overline{\theta}; a_1 = a^*(N) + \Delta)}{g^Q_1(S_1 \mid \overline{\theta}; a_1 = a^*(N))} X \mid a_1 = a^*(N)\right].
\end{aligned}$$

As this argument holds for arbitrary $\mathcal{F}^\infty$-measurable X, it must be that

$$\frac{d\mathcal{P}^{\Delta,N}}{d\mathcal{P}^{*N}} = \frac{g^Q_1(S_1 \mid \overline{\theta}; a_1 = a^*(N) + \Delta)}{g^Q_1(S_1 \mid \overline{\theta}; a_1 = a^*(N))}.$$

$\square$

To establish the desired limiting result, we will prove that for any Δ and N,

$$|\mu_N(\Delta) - \mu_\infty(\Delta)| \leq \kappa_N(\Delta) \frac{\beta}{\sqrt{N}},$$

where

$$\kappa_N(\Delta) \equiv \left(\mathbb{E}\left[\left(\frac{d\mathcal{P}^{\Delta,N}}{d\mathcal{P}^{*N}} - 1\right)^2 \,\middle|\, \mathbf{a} = \mathbf{a}^*(N)\right]\right)^{1/2}$$

and β is a finite constant independent of N and Δ whose value depends on the model. The following lemma establishes several important properties of κ_N.

Lemma C.2. *$\kappa_N(\Delta)$ is independent of N, $\kappa_N(0) = 0$, and $\overline{\kappa}'_{N,+}(0) = \limsup_{\Delta \downarrow 0} \kappa_N(\Delta)/\Delta < \infty$.*

Proof. We prove the theorem for the linked quality model, with nearly identical work establishing the result for the linked circumstance model. Note that when $\Delta = 0$, $d\mathcal{P}^{\Delta,N}/d\mathcal{P}^{*N} = 1$, and so trivially $\kappa_N(0) = 0$. To see that $\kappa_N(\Delta)$ is independent of N, note that the distribution of each outcome satisfies the translation invariance property $G^Q_i(S_i = s_i \mid \overline{\theta}; a_i = \alpha) =$

$G_i^Q(S_i = s_i - \alpha \mid \overline{\theta}; a_i = 0)$ for any s_i and α. So $\kappa_N(\Delta)$ may be written

$$\begin{aligned}
&\kappa_N(\Delta) \\
&= \int dF_{\overline{\theta}}(\overline{\theta})\, dG_1^Q(S_1 = s_1 \mid \theta; a_1 = a^*(N)) \left(\frac{g_1^Q(S_1 = s_1 \mid \overline{\theta}; a_1 = a^*(N) + \Delta)}{g_1^Q(S_1 = s_1 \mid \overline{\theta}; a_1 = a^*(N))} - 1 \right)^2 \\
&= \int dF_{\overline{\theta}}(\overline{\theta})\, dG_1^Q(S_1 = s_1 - a^*(N) \mid \overline{\theta}; a_1 = 0) \left(\frac{g_1^Q(S_1 = s_1 - a^*(N) \mid \overline{\theta}; a_1 = \Delta)}{g_1^Q(S_1 = s_1 - a^*(N) \mid \overline{\theta}; a_1 = 0)} - 1 \right)^2
\end{aligned}$$

So perform a change of variables to $s_1' \equiv s_1 - a^*(N)$ to obtain the representation

$$\kappa_N(\Delta) = \int dF_{\overline{\theta}}(\overline{\theta})\, dG_1^Q(S_1 = s_1' \mid \overline{\theta}; a_1 = 0) \left(\frac{g_1^Q(S_1 = s_1' \mid \overline{\theta}; a_1 = \Delta)}{g_1^Q(S_1 = s_1' \mid \overline{\theta}; a_1 = 0)} - 1 \right)^2,$$

which is independent of N, as desired.

Now, let $\xi \equiv \theta_1^\perp + \varepsilon_1$. Let f_ξ be the convolution of $f_{\theta^\perp}$ and f_ε. Then for any Δ, $g_1^Q(S_1 \mid \overline{\theta}; a_1 = a^*(N) + \Delta) = f_\xi(S_1 - \overline{\theta} - a^*(N) - \Delta) = f_\xi(\xi - \Delta)$ under the measure $\mathcal{P}^{*N}$. Hence

$$\begin{aligned}
\kappa_N(\Delta) &= \left(\mathbb{E}\left[\left(\frac{d\mathcal{P}^{\Delta,N}}{d\mathcal{P}^{*N}} - 1 \right)^2 \,\middle|\, \mathbf{a} = \mathbf{a}^*(N) \right] \right)^{1/2} \\
&= \left(\mathbb{E}\left[\left(\frac{f_\xi(\xi - \Delta)}{f_\xi(\xi)} - 1 \right)^2 \,\middle|\, \mathbf{a} = \mathbf{a}^*(N) \right] \right)^{1/2} \\
&= \int dF_\xi(\xi) \left(\frac{f_\xi(\xi - \Delta) - f_\xi(\xi)}{f_\xi(\xi)} \right)^2
\end{aligned}$$

We must therefore show that the limit

$$\begin{aligned}
\limsup_{\Delta \downarrow 0} \frac{1}{\Delta} \kappa(\Delta) &= \limsup_{\Delta \downarrow 0} \frac{1}{\Delta} \left(\int dF_\xi(\xi) \left(\frac{f_\xi(\xi - \Delta) - f_\xi(\xi)}{f_\xi(\xi)} \right)^2 \right)^{1/2} \\
&= \left(\limsup_{\Delta \downarrow 0} \int dF_\xi(\xi) \frac{1}{\Delta^2} \left(\frac{f_\xi(\xi - \Delta) - f_\varepsilon(\xi)}{f_\varepsilon(\xi)} \right)^2 \right)^{1/2}
\end{aligned}$$

exists and is finite. By Assumption 3, for Δ sufficiently close to 0 there exists a non-negative, integrable function $J(\cdot)$ such that

$$\frac{1}{\Delta^2} \left(\frac{f_\xi(\xi - \Delta) - f_\xi(\xi)}{f_\xi(\xi)} \right)^2 \leq J(\xi)$$

for all ξ. Then by reverse Fatou's lemma,

$$\begin{aligned}
\limsup_{\Delta \downarrow 0} \int dF_\xi(\xi) \frac{1}{\Delta^2} \left(\frac{f_\xi(\xi - \Delta) - f_\xi(\xi)}{f_\xi(\xi)} \right)^2 &\leq \int dF_\xi(\xi) \limsup_{\Delta \downarrow 0} \frac{1}{\Delta^2} \left(\frac{f_\xi(\xi - \Delta) - f_\xi(\xi)}{f_\xi(\xi)} \right)^2 \\
&\leq \int dF_\xi(\xi)\, J(\xi) < \infty,
\end{aligned}$$

as desired. $\square$

The bound on $|\mu_N(\Delta) - \mu_\infty(\Delta)|$ just claimed impies the desired result because for $\Delta > 0$ it may be rewritten

$$|(\mu_N(\Delta) - \mu)/\Delta - (\mu_\infty(\Delta) - \mu)/\Delta| \leq \frac{\kappa_N(\Delta) - \kappa_N(0)}{\Delta} \frac{\beta}{\sqrt{N}},$$

and thus by taking $\Delta \downarrow 0$ the inequality

$$|\mu'_N(0) - \mu'_\infty(0)| \leq \overline{\kappa}'_{N,+}(0) \frac{\beta}{N}$$

must hold. Then as $\overline{\kappa}'_{N,+}(0)$ is finite and independent of N, $\mu'_N(0) \to \mu'_\infty(0)$ as $N \to \infty$, as desired.

We now derive the claimed bound. To streamline notation, we will write $\mathbb{E}^{*N}$ to represent expectations conditioning on $\mathbf{a} = \mathbf{a}^*(N)$, and $\mathbb{E}^{\Delta,N}$ to represent expectations conditioning on $a_1 = a^*(N) + \Delta$ and $\mathbf{a}_{-1} = \mathbf{a}^*(N)$. Note first that the expected value of the principal's posterior estimate of θ_1 is a function only of the size of agent 1's distortion Δ, but *not* of the equilibrium action inference. Thus

$$\begin{aligned}
\mu_\infty(\Delta) &= \mathbb{E}[\mathbb{E}[\theta_1 \mid \mathbf{S}; \mathbf{a} = \mathbf{a}^*(\infty)] \mid \mathbf{a} = (a^*(\infty) + \Delta, \mathbf{a}^*(\infty))] \\
&= \mathbb{E}[\mathbb{E}[\theta_1 \mid \mathbf{S}; \mathbf{a} = \mathbf{a}^*(N)] \mid \mathbf{a} = (a^*(N) + \Delta, \mathbf{a}^*(N))] \\
&= \mathbb{E}^{\Delta,N}[\mathbb{E}^{*N}[\theta_1 \mid \mathbf{S}]].
\end{aligned}$$

So we may write

$$\mu_N(\Delta) - \mu_\infty(\Delta) = \mathbb{E}^{\Delta,N}[\mathbb{E}^{*N}[\theta_1 \mid \mathbf{S}_{1:N}] - \mathbb{E}^{*N}[\theta_1 \mid \mathbf{S}]].$$

Now, performing a change of measure,

$$\begin{aligned}
&\mathbb{E}^{\Delta,N}[\mathbb{E}^{*N}[\theta_1 \mid \mathbf{S}_{1:N}] - \mathbb{E}^{*N}[\theta_1 \mid \mathbf{S}]] \\
&= \mathbb{E}^{*N}\left[\frac{d\mathcal{P}^{\Delta,N}}{d\mathcal{P}^{*N}} \left(\mathbb{E}^{*N}[\theta_1 \mid \mathbf{S}_{1:N}] - \mathbb{E}^{*N}[\theta_1 \mid \mathbf{S}]\right)\right] \\
&= \mathbb{E}^{*N}\left[\left(\frac{d\mathcal{P}^{\Delta,N}}{d\mathcal{P}^{*N}} - 1\right) \left(\mathbb{E}^{*N}[\theta_1 \mid \mathbf{S}_{1:N}] - \mathbb{E}^{*N}[\theta_1 \mid \mathbf{S}]\right)\right] \\
&\quad + \mathbb{E}^{*N}[\mathbb{E}^{*N}[\theta_1 \mid \mathbf{S}_{1:N}] - \mathbb{E}^{*N}[\theta_1 \mid \mathbf{S}]] \\
&= \mathbb{E}^{*N}\left[\left(\frac{d\mathcal{P}^{\Delta,N}}{d\mathcal{P}^{*N}} - 1\right) \left(\mathbb{E}^{*N}[\theta \mid \mathbf{S}_{1:N}] - \mathbb{E}^{*N}[\theta_1 \mid \mathbf{S}]\right)\right],
\end{aligned}$$

with the last line following by the law of iterated expectations. Then by an application of the Cauchy-Schwarz inequality,

$$|\mu_N(\Delta) - \mu_\infty(\Delta)| \leq \kappa_N(\Delta) \left(\mathbb{E}^{*N}\left[\left(\mathbb{E}^{*N}[\theta_1 \mid \mathbf{S}_{1:N}] - \mathbb{E}^{*N}[\theta_1 \mid \mathbf{S}]\right)^2\right]\right)^{1/2}.$$

We will bound the rhs for the linked quality model, with the result for the linked circumstance model following by nearly identical work.

Define the family of random variables $\widehat{\theta}_N(z) \equiv \mathbb{E}^{*N}[\theta_1 \mid S_1, \overline{\theta} = z]$ for $z \in \mathbb{R}$. Note that $\widehat{\theta}_1(\overline{\theta}) = \mathbb{E}^{*N}[\theta_1 \mid \mathbf{S}]$, as $\mathbf{S}$ allows the principal to perfectly infer $\overline{\theta}$, and θ_1 is independent of the vector of outcomes $\mathbf{S}_{-1}$ conditional on $\overline{\theta}$. Further, $\mathbb{E}^{*N}[\theta_1 \mid \mathbf{S}_{1:N}] = \mathbb{E}^{*N}[\mathbb{E}^{*N}[\theta_1 \mid \mathbf{S}] \mid \mathbf{S}_{1:N}]$ is the mean-square minimizing estimator of $\widehat{\theta}_N(\overline{\theta})$ conditional on the performance vector $\mathbf{S}_{1:N}$. Another estimator of $\widehat{\theta}_N(\overline{\theta})$ is $\widehat{\theta}_N\left(\widetilde{\theta}_N\right)$, where

$$\widetilde{\theta}_N \equiv \frac{1}{N}\sum_{i=1}^{N}(S_i - \mu^\perp),$$

with $\mu^\perp = \mathbb{E}[\theta_i^\perp]$. So

$$\mathbb{E}^{*N}\left[\left(\mathbb{E}^{*N}[\theta_1 \mid \mathbf{S}_{1:N}] - \mathbb{E}^{*N}[\theta_1 \mid \mathbf{S}]\right)^2\right] \leq \mathbb{E}^{*N}\left[\left(\widehat{\theta}_N\left(\widetilde{\theta}_N\right) - \mathbb{E}^{*N}[\theta_1 \mid \mathbf{S}]\right)^2\right].$$

Given that shifts in $\overline{\theta}$ affect the outcome S_i additively, $\mathbb{E}^{*N}[\theta_1 \mid S_1 = s_1, \overline{\theta} = z] = \mathbb{E}^{*N}[\theta_1 \mid S_1 = s_1 - z, \overline{\theta} = 0]$ for every s_1 and z. The proof of Lemma B.9 establishes that $\mathbb{E}^{*N}[\theta_1 \mid S_1, \overline{\theta}]$ is differentiable wrt S_1 and uniformly bounded in $(0,1)$ everywhere. Hence $\widehat{\theta}_N(z)$ is differentiable and $\widehat{\theta}'_N(z) \in (-1, 0)$ for all z. Thus by the fundamental theorem of calculus,

$$|\widehat{\theta}_N(\widetilde{\theta}_N) - \widehat{\theta}_N(\overline{\theta})| = \left|\int_{\overline{\theta}}^{\widetilde{\theta}_N} \widehat{\theta}'_N(z)\, dt\right| \leq \int_{\overline{\theta}}^{\widetilde{\theta}_N} |\widehat{\theta}'_N(z)|\, dz \leq |\widetilde{\theta}_N - \overline{\theta}|.$$

Further note that

$$\widetilde{\theta}_N - \overline{\theta} = \frac{1}{N}\sum_{i=1}^{N}(\theta_i^\perp - \mu^\perp + \varepsilon_i),$$

which has mean 0 and variance $(\sigma^2_{\theta^\perp} + \sigma^2_\varepsilon)/N$ given that $\theta_i^\perp$ and ε_i are mutually independent. So

$$\mathbb{E}^{*N}\left[\left(\mathbb{E}^{*N}[\theta_1 \mid \mathbf{S}_{1:N}] - \mathbb{E}^{*N}[\theta_1 \mid \mathbf{S}]\right)^2\right] \leq \frac{\sigma^2_{\theta^\perp} + \sigma^2_\varepsilon}{N},$$

implying the desired bound with $\beta = \sqrt{\sigma^2_{\theta^\perp} + \sigma^2_\varepsilon}$.

D Proofs for Section 4 (Main Results)

D.1 Proofs of Theorems 1 and 2

Opt-In Equilibrium. In any pure-strategy equilibrium in which all agents opt-in, the equilibrium effort level a^* must satisfy two conditions:

$$MV(N) = C'(a^*) \tag{D.1}$$

$$R + \mu - C(a^*) \geq 0 \tag{D.2}$$

The expression in (D.1) guarantees that an agent who opts-in cannot strictly gain by deviating to a different effort choice. This is identical to the condition used in the exogenous entry model to solve for equilibrium. The expression in (D.2) guarantees that agents cannot profitably deviate to opting-out.

The marginal value $MV(N)$ is independent of a^*, and C' is strictly monotone. Thus (D.1) pins down a unique effort level $a^* = C'^{-1}(MV(N))$. Since C is everywhere increasing, the conditions in (D.1) and (D.2) can be simultaneously satisfied if and only if $0 \leq C'^{-1}[MV(N)] \leq a^{**} \equiv C^{-1}(R + \mu)$, or equivalently,

$$0 = C'(0) \leq MV(N) \leq C'(a^{**})$$

noting that C'^{-1} is everywhere increasing.

By Assumption 6, $R + \mu > C(a^*(1))$. Since the cost function C has positive first and second derivatives, $R + \mu > C(a^*(1))$ and $R + \mu = C(a^{**})$ imply that $a^*(1) < a^{**}$, which further implies $C'(a^*(1)) < C'(a^{**})$. By Lemma 1, $MV(1) = MV_Q(1) \geq MV_Q(N)$. Thus

$$MV_Q(N) \leq MV_Q(1) = C'(a^*(1)) \leq C'(a^{**}),$$

and a symmetric all opt-in equilibrium exists in the linked quality model. In contrast, in the linked circumstance model,

$$MV_C(N) \geq MV_C(1) = C'(a^*(1)) \tag{D.3}$$

so the inequality $MV_C(N) \leq C'(a^{**})$ is not guaranteed to hold. An opt-in equilibrium exists if and only if N is sufficiently small; specifically, $N \leq N^*$ where

$$N^* \equiv \sup\{N \ : \ MV_C(N) \leq C'(a^{**})\}.$$

(It is possible that N^* is infinite if $MV_C(N) \leq C'(a^{**})$ for all N.)

Finally, for the parameters $N \leq N^*$ where an opt-in equilibrium exists in both models, it is possible to rank equilibrium effort levels as follows: Define a_C^* and a_Q^* to be the respective equilibrium effort levels. Then, since $MV_C(N) \geq MV(1) \geq MV_Q(N)$ for all N,

$$a_C^* = C'^{-1}(MV_C(N)) \geq C^{-1}(MV_Q(N)) = a_Q^*$$

so equilibrium effort is higher in the linked circumstance model.

Opt-Out Equilibrium. Under the imposed refinement on the principal's off-equilibrium belief about the agent's action, the optimal action conditional on entry is $a^*(1)$. Thus in an all opt-out equilibrium, the equilibrium action a^* must satisfy

$$R + \mu - C(a^*(1)) < 0 \tag{D.4}$$

which violates Assumption 6. There are no pure-strategy equilibria in either model in which all agents choose to opt-out.

Mixed Equilibrium. For any probability $p \in [0, 1]$ and $M \in \{T, C\}$, let

$$MV_M(p, N) = \mathbb{E}\left[\left(MV_M(\widetilde{N} + 1\right) \mid \widetilde{N} \sim \text{Binomial}(N - 1, p)\right]$$

be the expected marginal impact for agent i of exerting additional effort beyond the principal's expectation, when agent i opts-in and all other agents opt-in with independent probability p. Note that because $MV_C(N)$ is increasing in N, and increasing p shifts up the distribution of $\widetilde{N}$ in the FOSD sense, $MV_C(p, N)$ is increasing in p. Further, because increasing p shifts $\Pr(\widetilde{N} \leq n)$ strictly downward for every $n < N - 1$, this monotonicity is strict whenever $MV_C(n)$ is not constant over the range $\{1, .., N\}$. For the same reasons, $MV_C(p, N)$ is increasing in N for fixed p, and strictly increasing whenever $p \in (0, 1)$ and $MV_C(n)$ is not constant over $\{1, ..., N\}$.

In a mixed equilibrium, the equilibrium effort level a^* and probability p assigned to opting-in must jointly satisfy

$$R + \mu - C(a^*) = 0. \tag{D.5}$$

$$MV(p, N) = C'(a^*). \tag{D.6}$$

The expression in (D.5) pins down the equilibrium action, which is identical to the action defined as a^{**} above. Moreover, $C'(a)$ is independent of both the mixing probability p and also the fixed segment size N. Therefore an equilibrium exists if and only if $MV(p, N) = C'(a^{**})$ for some $p \in [0, 1]$. But for all $p \in [0, 1]$,

$$MV_Q(p, N) \leq \max_{1 \leq N' \leq N} MV_Q(N') = MV_Q(1) = C'(a^*(1)) < C'(a^{**})$$

using that MV_Q is a decreasing function of N (Lemma 1). Thus the linked quality model does not admit a strictly mixed equilibrium.

Similarly if $MV_C(N) < C'(a^{**})$, then

$$MV_C(p, N) \leq \max_{1 \leq N' \leq N} MV_C(N') = MV_C(N) < C'(a^{**})$$

since MV_C is a strictly increasing function of N (Lemma 1). So there does not exist a strictly mixed equilibrium in the linked circumstance model either. Indeed, this is exactly the range for N that supports the symmetric all opt-in equilibrium in the linked circumstance model.

If however $MV(N) \geq C'(a^{**})$, then

$$MV_C(1) = MV_C(0, N) < C'(a^{**}) \leq MV_C(1, N) = MV_C(N).$$

This implies in particular that MV_C is not constant over the range $\{1, ..., N\}$, so that $MV_C(p, N)$ is strictly increasing in p. Since $MV_C(p, N)$ is also continuous in p, the intermediate value theorem yields existence of a unique $p^*(N) \in (0, 1]$ satisfying $MV_C(p^*(N), N) = C'(a^{**})$.

If $N \leq N^*$, i.e. $MV(N) = C'(a^{**})$, then it must be that $p^*(N) = 1$. Thus in particular the opt-in equilibrium is unique whenever it exists. Otherwise $p^*(N) < 1$, in which case the fact that $MV_C(p, N)$ is strictly increasing in N for fixed $p \in (0, 1)$ further implies that $p^*(N)$ must be strictly decreasing in N. Finally, the effort level a^{**} chosen in this equilibrium weakly exceeds the effort level a_C^* chosen in the symmetric opt-in equilibrium in the linked circumstance model, since $R + \mu \geq C(a_C^*)$ by (D.2), while $R + \mu = C(a^{**})$ by (D.5).

D.2 Proof of Proposition 1

Comparisons between equilibrium actions correspond directly to comparisons of marginal values of effort. It is therefore sufficient to establish that $MV(N) < 1$ for all N, and that $MV_Q(N)$ is decreasing while $MV_C(N)$ is increasing in N, with $\lim_{N\to\infty} MV_C(N) < 1$. These facts in particular imply that $MV_Q(N) \leq MV(1) \leq MV_C(N)$, with $MV(1)$ dictating equilibrium effort in the no-sharing benchmark. Lemma 1 establishes the desired monotonicity of the marginal value of effort, while the upper bound on MV and the limiting value of MV_C are established in Appendix B.2.

D.3 Proof of Proposition 2

Suppose all agents in a segment of size N enter and choose action a. Social welfare

$$W(1, a, N) = N \cdot (2\mu + a - C(a))$$

is strictly increasing on $a \in [0, a_{FB})$. Thus the comparison $a_Q^*(N) \leq a_{NDL} < a_{FB}$ immediately implies that for all N, welfare is ranked

$$W_Q(N) \leq W_{NDL}(N)$$

where the inequality is strict for all $N > 1$.

For segment sizes $N < N^*$, the equilibrium action in the linked circumstance model satisfies $a_C^*(N) \in [a_{NDL}, a_{FB})$ (Theorem 2), so the same argument implies

$$W_{NS}(N) \leq W_C(N)$$

with the inequality strict for $N > 1$. When the segment size $N > N^*$,

$$W_C(N) = N \cdot p(N) \cdot [a^{**} - C(a^{**}) + 2\mu] .$$

Since $p(N) \to 0$ as $N \to \infty$, it follows that for N sufficiently large,

$$W_C(N), W_Q(N) < W_{NDL}(N).$$

E Proofs for Gaussian Case

E.1 Verification of Assumptions in 2.5

Here we verify that Gaussian uncertainty satisfies the stated assumptions. Assumptions 1, 2, and 4 are immediate. Assumption 5 is satisfied for any strictly convex cost function, since the second derivative of the posterior expectation in each signal realization is zero. Assumption 3 is verified in the following lemma:

Lemma E.1. *Suppose $\xi \sim \mathcal{N}(0, \sigma^2)$. Then for any $\overline{\Delta} > 0$, the function*

$$J^*(\xi) = \frac{1}{\overline{\Delta}^2} \left(\exp\left(\frac{\overline{\Delta}^2}{2\sigma^2} \right) + \exp\left(\frac{\overline{\Delta}|\xi|}{\sigma^2} \right) - 2 \right)^2$$

satisfies $|J(\xi, \Delta)| \leq J^*(\xi)$ for every $\xi \in \mathbb{R}$ and $\Delta \in [-\overline{\Delta}, \overline{\Delta}]$, and $\mathbb{E}[J^*(\xi)] < \infty$.

Proof. Under the distributional assumption on ξ, the density function f_ξ has the form

$$f_\xi(\xi) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{\xi^2}{2\sigma^2} \right).$$

Therefore

$$\frac{1}{\Delta} \frac{f_\xi(\xi - \Delta) - f_\xi(\xi)}{f_\xi(\xi)} = \frac{\exp\left(\frac{1}{\sigma^2}\Delta(\xi - \Delta/2)\right) - 1}{\Delta}.$$

Now, we may equivalently write

$$\begin{aligned}\frac{1}{\Delta}\frac{f_\xi(\xi-\Delta)-f_\xi(\xi)}{f_\xi(\xi)} &= \frac{1}{\sigma^2}\int_{\Delta/2}^{\xi}\exp\left(\frac{1}{\sigma^2}\Delta(\widetilde{\xi}-\Delta/2)\right)\,d\widetilde{\xi} \\ &= \frac{\exp\left(-\frac{\Delta^2}{2\sigma^2}\right)}{\sigma^2}\int_{\Delta/2}^{\xi}\exp\left(\frac{\Delta\widetilde{\xi}}{\sigma^2}\right)\,d\widetilde{\xi}.\end{aligned}$$

Hence

$$\begin{aligned}\left|\frac{1}{\Delta}\frac{f_\xi(\xi-\Delta)-f_\xi(\xi)}{f_\xi(\xi)}\right| &= \frac{\exp\left(-\frac{\Delta^2}{2\sigma^2}\right)}{\sigma^2}\int_{\min\{\Delta/2,\xi\}}^{\max\{\Delta/2,\xi\}}\exp\left(\frac{\Delta\widetilde{\xi}}{\sigma^2}\right)\,d\widetilde{\xi} \\ &\leq \frac{1}{\sigma^2}\int_{\min\{\Delta/2,\xi\}}^{\max\{\Delta/2,\xi\}}\exp\left(\frac{\Delta\widetilde{\xi}}{\sigma^2}\right)\,d\widetilde{\xi}.\end{aligned}$$

Let

$$H(\xi,\Delta)\equiv\frac{1}{\sigma^2}\int_{\min\{\Delta/2,\xi\}}^{\max\{\Delta/2,\xi\}}\exp\left(\frac{\Delta\widetilde{\xi}}{\sigma^2}\right)\,d\widetilde{\xi}.$$

We will show that $H(\xi,\Delta)\leq\sqrt{J^*(\xi)}$ for all ξ and $\Delta\in[-\overline{\Delta},\overline{\Delta}]$ in cases, depending on the signs of ξ,Δ, and $\xi-\Delta/2$.

Case 1: $\xi\geq\Delta/2\geq 0$. Then

$$\begin{aligned}H(\xi,\Delta) &= \frac{1}{\sigma^2}\int_{\Delta/2}^{\xi}\exp\left(\frac{\Delta\widetilde{\xi}}{\sigma^2}\right)\,d\widetilde{\xi} \\ &\leq \frac{1}{\sigma^2}\int_{0}^{\xi}\exp\left(\frac{\overline{\Delta}\widetilde{\xi}}{\sigma^2}\right)\,d\widetilde{\xi} \\ &= \frac{1}{\overline{\Delta}}\left(\exp\left(\frac{\overline{\Delta}\xi}{\sigma^2}\right)-1\right)\leq\sqrt{J^*(\xi)}.\end{aligned}$$

Case 2: $\xi\geq 0>\Delta/2$. Then

$$\begin{aligned}H(\xi,\Delta) &= \frac{1}{\sigma^2}\int_{\Delta/2}^{\xi}\exp\left(\frac{\Delta\widetilde{\xi}}{\sigma^2}\right)\,d\widetilde{\xi} \\ &\leq \frac{1}{\sigma^2}\left(\int_{0}^{\xi}\exp\left(\frac{\overline{\Delta}\widetilde{\xi}}{\sigma^2}\right)\,d\widetilde{\xi}+\int_{-\overline{\Delta}/2}^{0}\exp\left(-\frac{\overline{\Delta}\widetilde{\xi}}{\sigma^2}\right)\,d\widetilde{\xi}\right) \\ &= \frac{1}{\overline{\Delta}}\left(\exp\left(\frac{\overline{\Delta}\xi}{\sigma^2}\right)+\exp\left(\frac{\overline{\Delta}^2}{2\sigma^2}\right)-2\right) \\ &= \sqrt{J^*(\xi)}.\end{aligned}$$

Case 3: $\Delta/2 > \xi \geq 0$. Then

$$\begin{aligned}
H(\xi, \Delta) &= \frac{1}{\sigma^2} \int_{\xi}^{\Delta/2} \exp\left(\frac{\Delta\widetilde{\xi}}{\sigma^2}\right) d\widetilde{\xi} \\
&\leq \frac{1}{\sigma^2} \int_{0}^{\overline{\Delta}/2} \exp\left(\frac{\overline{\Delta}\widetilde{\xi}}{\sigma^2}\right) d\widetilde{\xi} \\
&= \frac{1}{\overline{\Delta}} \left(\exp\left(\frac{\overline{\Delta}^2}{2\sigma^2}\right) - 1\right) \\
&\leq \sqrt{J^*(\xi)}.
\end{aligned}$$

Case 4: $\Delta/2 > 0 > \xi$. Then

$$\begin{aligned}
H(\xi, \Delta) &= \frac{1}{\sigma^2} \int_{\xi}^{\Delta/2} \exp\left(\frac{\Delta\widetilde{\xi}}{\sigma^2}\right) d\widetilde{\xi} \\
&\leq \frac{1}{\sigma^2} \left(\int_{0}^{\overline{\Delta}/2} \exp\left(\frac{\overline{\Delta}\widetilde{\xi}}{\sigma^2}\right) d\widetilde{\xi} + \int_{\xi}^{0} \exp\left(-\frac{\overline{\Delta}\widetilde{\xi}}{\sigma^2}\right) d\widetilde{\xi}\right) \\
&= \frac{1}{\overline{\Delta}} \left(\exp\left(\frac{\overline{\Delta}^2}{2\sigma^2}\right) + \exp\left(\frac{\overline{\Delta}|\xi|}{\sigma^2}\right) - 2\right) \\
&= \sqrt{J^*(\xi)}.
\end{aligned}$$

Case 5: $0 \geq \Delta/2 > \xi$. Then

$$\begin{aligned}
H(\xi, \Delta) &= \frac{1}{\sigma^2} \int_{\xi}^{\Delta/2} \exp\left(\frac{\Delta\widetilde{\xi}}{\sigma^2}\right) d\widetilde{\xi} \\
&\leq \frac{1}{\sigma^2} \int_{\xi}^{0} \exp\left(-\frac{\overline{\Delta}\widetilde{\xi}}{\sigma^2}\right) d\widetilde{\xi} \\
&= \frac{1}{\overline{\Delta}} \left(\exp\left(\frac{\overline{\Delta}|\xi|}{\sigma^2}\right) - 1\right) \leq \sqrt{J^*(\xi)}.
\end{aligned}$$

Case 6: $0 > \xi \geq \Delta/2$. Then

$$\begin{aligned}
H(\xi, \Delta) &= \frac{1}{\sigma^2} \int_{\Delta/2}^{\xi} \exp\left(\frac{\Delta\widetilde{\xi}}{\sigma^2}\right) d\widetilde{\xi} \\
&\leq \frac{1}{\sigma^2} \int_{-\overline{\Delta}/2}^{0} \exp\left(-\frac{\overline{\Delta}\widetilde{\xi}}{\sigma^2}\right) d\widetilde{\xi} \\
&= \frac{1}{\overline{\Delta}} \left(\exp\left(\frac{\overline{\Delta}^2}{2\sigma^2}\right) - 1\right) \leq \sqrt{J^*(\xi)}.
\end{aligned}$$

This establishes that $|J(\xi,\Delta)| \le H(\xi,\Delta)^2 \le J^*(\xi)$ for every ξ and $\Delta \in [-\overline{\Delta},\overline{\Delta}]$, as desired. It remains only to show that J^* is $\mathcal{P}^0$-integrable. This follows because

$$\begin{aligned}
J^*(\xi) &\le \frac{1}{\overline{\Delta}^2}\left(\exp\left(\frac{\overline{\Delta}^2}{2\sigma^2}\right) + \exp\left(\frac{\overline{\Delta}|\xi|}{\sigma^2}\right)\right)^2 \\
&= \frac{1}{\overline{\Delta}^2}\left(\exp\left(\frac{\overline{\Delta}^2}{\sigma^2}\right) + 2\exp\left(\frac{\overline{\Delta}^2}{\sigma^2}\right)\exp\left(\frac{\overline{\Delta}|\xi|}{\sigma^2}\right) + \exp\left(\frac{2\overline{\Delta}|\xi|}{\sigma^2}\right)\right) \\
&= \frac{1}{\overline{\Delta}^2}\left(\exp\left(\frac{\overline{\Delta}^2}{\sigma^2}\right) + 2\exp\left(\frac{\overline{\Delta}^2}{\sigma^2}\right)\left(\exp\left(\frac{\overline{\Delta}\xi}{\sigma^2}\right) + \exp\left(-\frac{\overline{\Delta}\xi}{\sigma^2}\right)\right)\right. \\
&\qquad\qquad \left. + \exp\left(\frac{2\overline{\Delta}\xi}{\sigma^2}\right) + \exp\left(-\frac{2\overline{\Delta}\xi}{\sigma^2}\right)\right)
\end{aligned}$$

The first term is a constant, while each of the remaining terms is proportional to a lognormal random variable. Thus each term has finite mean, and hence so does $J^*(\xi)$. □

E.2 Marginal Value of Effort

Consider the linked quality model, and suppose that agent i chooses effort $a_i = a^* + \Delta$ while all agents $j \neq i$ choose the equilibrium effort level a^*. The principal's posterior belief about $\overline{\theta} + \theta_i^{\perp}$ is independent of $\mathbf{S}_{-i}$ conditional on $\overline{\theta}$. Thus, using standard formulas for updating to normal signals, we can first update the principal's belief about $\overline{\theta}$ to

$$\overline{\theta} \mid \mathbf{S}_{-i} \sim \mathcal{N}\left(\frac{(N-1)\sigma_{\overline{\theta}}^2 \cdot (\overline{S}_{-i} - a^*) + (\sigma_{\theta^\perp}^2 + \sigma_\varepsilon^2)\cdot\mu}{(N-1)\sigma_{\overline{\theta}}^2 + \sigma_{\theta^\perp}^2 + \sigma_\varepsilon^2}, \frac{\sigma_{\overline{\theta}}^2}{(N-1)\sigma_{\overline{\theta}^2} + \sigma_{\theta^\perp}^2 + \sigma_\varepsilon^2}\right) \equiv \mathcal{N}\left(\hat{\mu}_{\overline{\theta}}, \hat{\sigma}_{\overline{\theta}^2}\right)$$

where $\overline{S}_{-i}$ is the average outcome. The principal's expectation of $\overline{\theta}+\theta_i^\perp$ after further updating to S_i is

$$\mathbb{E}(\overline{\theta} + \theta_i^\perp \mid \mathbf{S}) = \frac{\sigma_\varepsilon^2}{\hat{\sigma}_{\overline{\theta}^2} + \sigma_{\theta^\perp}^2 + \sigma_\varepsilon^2}\cdot(\overline{S}_{-i} - a^*) + \frac{\hat{\sigma}_{\overline{\theta}^2} + \sigma_{\theta^\perp}^2}{\hat{\sigma}_{\overline{\theta}^2} + \sigma_{\theta^\perp}^2 + \sigma_\varepsilon^2}\cdot(S_i - a^*).$$

Taking an expectation with respect to the agent's prior belief, we have:

$$\begin{aligned}
\mu_N(\Delta) = \mathbb{E}\left[\mathbb{E}(\overline{\theta} + \theta_i^\perp \mid S)\right] &= \frac{\sigma_\varepsilon^2}{\hat{\sigma}_{\overline{\theta}^2} + \sigma_{\theta^\perp}^2 + \sigma_\varepsilon^2}\cdot\mu + \frac{\hat{\sigma}_{\overline{\theta}^2} + \sigma_{\theta^\perp}^2}{\hat{\sigma}_{\overline{\theta}^2} + \sigma_{\theta^\perp}^2 + \sigma_\varepsilon^2}\cdot(\mu + \Delta) \\
&= \mu + \frac{\hat{\sigma}_{\overline{\theta}^2} + \sigma_{\theta^\perp}^2}{\hat{\sigma}_{\overline{\theta}^2} + \sigma_{\theta^\perp}^2 + \sigma_\varepsilon^2}\cdot\Delta
\end{aligned}$$

and the marginal value of effort is

$$\begin{aligned}
\mu_N'(\Delta) &= \frac{\hat{\sigma}_{\overline{\theta}^2} + \sigma^2_{\theta^\perp}}{\hat{\sigma}_{\overline{\theta}^2} + \sigma^2_{\theta^\perp} + \sigma^2_\varepsilon} \\
&= \left(\frac{\sigma^2_{\overline{\theta}}}{(N-1)\sigma_{\overline{\theta}^2} + \sigma^2_{\theta^\perp} + \sigma^2_\varepsilon} + \sigma^2_{\theta^\perp} \right) \Big/ \left(\frac{\sigma^2_{\overline{\theta}}}{(N-1)\sigma_{\overline{\theta}^2} + \sigma^2_{\theta^\perp} + \sigma^2_\varepsilon} + \sigma^2_{\theta^\perp} + \sigma^2_\varepsilon \right). \quad \text{(E.1)}
\end{aligned}$$

It is straightforward to verify that this expression is independent of Δ, decreasing in N, and converges to $\sigma^2_{\theta^\perp} / \left(\sigma^2_{\theta^\perp} + \sigma^2_\varepsilon\right)$ as $N \to \infty$.

Consider now the linked circumstance model. Using parallel arguments to those above, the principal's posterior belief about the common part of the noise shock $\overline{\varepsilon}$ after updating to $\mathbf{S}_{-i}$ is

$$\overline{\varepsilon} \mid \mathbf{S}_{-i} \sim \mathcal{N}\left(\frac{(N-1)\sigma^2_{\overline{\varepsilon}}}{(N-1)\sigma^2_{\overline{\varepsilon}} + \sigma^2_\theta + \sigma^2_{\varepsilon^\perp}} \cdot \left(\overline{S}_{-i} - a^* - \mu\right), \frac{\sigma^2_{\overline{\varepsilon}}(\sigma^2_{\varepsilon^\perp} + \sigma^2_\theta)}{(N-1)\sigma^2_{\overline{\varepsilon}} + \sigma^2_{\varepsilon^\perp} + \sigma^2_\theta} \right) \equiv \mathcal{N}(\eta, \hat{\sigma}^2_{\overline{\varepsilon}})$$

and the principal's posterior expectation of θ_i after further updating to S_i is

$$\mathbb{E}(\theta_i \mid \mathbf{S}) = \frac{\sigma^2_\theta}{\sigma^2_\theta + \hat{\sigma}^2_{\overline{\varepsilon}} + \sigma^2_{\varepsilon^\perp}} \cdot (S_i - \eta) + \frac{\hat{\sigma}^2_{\overline{\varepsilon}} + \sigma^2_{\varepsilon^\perp}}{\sigma^2_\theta + \hat{\sigma}^2_{\overline{\varepsilon}} + \sigma^2_{\varepsilon^\perp}} \cdot \mu$$

Since in the agent's prior, $\mathbb{E}(S_i) = \mu + \Delta$ and $\mathbb{E}(\eta) = 0$, the agent's expectation of the principal's forecast is

$$\mu_N(\Delta) = \mathbb{E}(\theta_i \mid \mathbf{S}) = \mu + \frac{\sigma^2_\theta}{\sigma^2_\theta + \hat{\sigma}^2_{\overline{\varepsilon}} + \sigma^2_{\varepsilon^\perp}} \cdot \Delta$$

implying that the marginal value of effort is

$$\begin{aligned}
\mu_N'(\Delta) &= \sigma^2_\theta / (\sigma^2_\theta + \hat{\sigma}^2_{\overline{\varepsilon}} + \sigma^2_\varepsilon) \\
&= \sigma^2_\theta \Big/ \left(\sigma^2_\theta + \frac{\sigma^2_{\overline{\varepsilon}}(\sigma^2_{\varepsilon^\perp} + \sigma^2_\theta)}{(N-1)\sigma^2_{\overline{\varepsilon}} + \sigma^2_{\varepsilon^\perp} + \sigma^2_\theta} + \sigma^2_\varepsilon \right) \qquad \text{(E.2)}
\end{aligned}$$

This expression is constant in Δ, increasing in N, and converges to $\sigma^2_\theta/(\sigma^2_\theta + \sigma^2_\varepsilon)$ as N grows large.

E.3 Expected Number of Entrants in Mixed Equilibrium

It follows from the expressions in (E.1) and (E.2) that:

Lemma E.2. *(a) $MV_Q(N)$ is everywhere convex.*

(b) $MV_C(N)$ is everywhere concave.

Thus, uncertainty about the number of entering agents has a dampening effect on incentives in the linked circumstance model but strengthens the effect on incentives in the linked quality model.

Additionally, Lemma E.2 allows us to make stronger statements about the probability of entry in the mixed equilibrium. Let $p^*(N)$ be the opt-in fraction such that given *deterministic* entry of $p^*(N) \cdot N$ agents, then equilibrium effort is a^{**} (see (4)). Then:

Proposition E.1. *Fix any $N > N^*$. In the unique symmetric equilibrium of the linked circumstance model:*

(a) The expected number of agents opting in, $p(N) \cdot N$, is strictly increasing in N.

(b) The probability of entry $p(N)$ strictly exceeds $p^*(N)$.

(c) The equilibrium effort a^{**} is less than the equilibrium effort $a^*_C[p(N) \cdot N]$ associated with full-entry in a deterministic segment of size $p(N) \cdot N$.

Part (a) of Proposition E.1 says that even though the probability of entry decreases in N, the *expected* number of entrants $p(N) \cdot N$ is increasing in the total segment size. Part (b) says that in equilibrium, the probability of entry $p(N)$ exceeds $p^*(N)$. Thus, uncertainty in the number of entrants has the effect of *increasing* the number of expected entrants. Since any profile in which exactly $p^*(N) \cdot N$ agents opt-in and choose equilibrium effort a^{**} is an asymmetric equilibrium, this further implies that there is strictly less entry in those pure-strategy asymmetric equilibria than in the unique symmetric mixed equilibrium. Finally, part (c) says that the equilibrium effort in this mixed equilibrium is smaller than the effort level that would be chosen given a deterministic segment of size $p(N) \cdot N$. That is, uncertainty in the number of entrants has the effect of decreasing effort.

Proof. Parts (b) and (c) follow directly from concavity of $MV_C(N)$ (Lemma E.2), since

$$MV(p(N) \cdot N) > \mathbb{E}[MV(\widetilde{N} \sim \mathrm{Bin}(N, p(N)))] = MV(p^*(N) \cdot N)$$

and $MV(N)$ is increasing in its argument.

We now show Part (a), which says that the expected number of entrants $p(N) \cdot N$ is increasing in N. With N total agents, the number of entrants in the unique symmetric equilibrium is distributed as $\mathrm{Bin}(N, p(N))$. Fix N, and let $q \equiv p(N)$. Consider increasing the number of agents to $N' > N$, and let $q' \equiv p(N)N/N'$, so that $\mathrm{Bin}(N, q)$ and $\mathrm{Bin}(N', q')$ have the same mean. By Lemma E.2, $MV_C(N)$ is strictly concave in N. So let $\widetilde{N} \sim \mathrm{Bin}(N, q)$ and

$\widetilde{N}' \sim \text{Bin}(N', q')$. If $\widetilde{N}'$ is a mean-preserving spread of $\widetilde{N}$, then $\mathbb{E}[MV_C(\widetilde{N}')] < \mathbb{E}[MV_C(\widetilde{N})]$, meaning that

$$MV_C(p(N'), N') = C'(a^{**}) = \mathbb{E}[MV_C(\widetilde{N})] > \mathbb{E}[MV_C(\widetilde{N}')] = MV_C(q', N')$$

and therefore $p(N') > q'$ given that $MV_C(\cdot, N)$ is a strictly increasing function. Then since $q'N' = qN$, the mean number of entrants $p(N')N'$ with N' agents is strictly larger than the mean number $p(N)N$ with N agents. So the theorem is proven once we establish that $\widetilde{N}$ is a mean-preserving spread of N.

Because the mean-preserving spread property is transitive, it is sufficient to prove that for every positive integer N and $q \in (0,1)$, the random variable $\widetilde{X} \sim \text{Bin}(N+1, q')$ with $q' \equiv qN/(N+1)$ is a mean-preserving spread of the random variable $X \sim \text{Bin}(N, q)$. Define the CDFs F and $\widetilde{F}$ for X and $\widetilde{X}$, respectively, satisfying

$$F(x) = \begin{cases} \sum_{i=0}^{\lfloor x \rfloor} \binom{N}{i} q^i (1-q)^{N-i}, & 0 \le x \le N \\ 0, & x < 0 \\ 1, & x > N \end{cases}$$

and

$$\widetilde{F}(x) = \begin{cases} \sum_{i=0}^{\lfloor x \rfloor} \binom{N+1}{i} (q')^i (1-q')^{N+1-i}, & 0 \le x \le N+1 \\ 0, & x < 0 \\ 1, & x > N+1 \end{cases}$$

Also define

$$\Gamma(x) \equiv \int_{-\infty}^{C} (\widetilde{F}(t) - F(t))\, dt.$$

In order that $\widetilde{X}$ be a mean-preserving spread of X it must be that $\Gamma(x) \ge 0$ for all x. For $x \le 0$ trivially $\Gamma(x) = 0$, and meanwhile for any distribution function G with mean μ and support contained in $(-\infty, \overline{x}]$,

$$\int_{-\infty}^{\overline{x}} G(t)\, dt = \int_{-\infty}^{\overline{x}} \int_{-\infty}^{C} dG(s)\, dt = \int_{-\infty}^{\overline{x}} dG(s) \int_{s}^{\overline{x}} dt = \int_{-\infty}^{\overline{x}} (\overline{x} - s)\, dG(s) = \overline{x} - \mu.$$

Hence $\Gamma(x) = 0$ for all $x \ge N+1$ as well. It therefore remains only to establish the result for $x \in (0, N+1)$.

Differentiating Γ yields

$$\Gamma'(x) = \widetilde{F}(x) - F(x).$$

Suppose that Γ' satisfies single-crossing on $[0, N+1]$; that is, there exists an $x_0 \in (0, N+1)$ such that $\Gamma'(x) > 0$ for $x < x_0$ and $\Gamma'(x) < 0$ for $x > x_0$. Then Γ is single-peaked on

$[0, N+1]$, and given that $\Gamma(0) = \Gamma(N+1) = 0$, it must therefore be that $\Gamma(x) > 0$ for all $x \in (0, N+1)$. We complete the proof by establishing single-crossing.

First note that for $x \in [N, N+1)$, $\Gamma'(x) = \widetilde{F}(N) - 1 < 0$. So it is sufficient to establish single-crossing on $[0, N]$. As a preliminary step, note that

$$\Gamma'(0) = (1-q)^N - (1-q')^{N+1} = \xi(q)^{Nq} - \xi(q')^{(N+1)q'} = \xi(q)^{Nq} - \xi(q')^{Nq},$$

where $\xi(x) \equiv (1 + 1/x)^x$. This function is well-known to be strictly increasing in x, so $\Gamma'(0) > 0$. Then as $\Gamma'(N) < 0$, Γ' must cross zero an odd number of times. We complete the proof by showing that Γ' can cross zero at most twice, establishing that it must cross exactly once, as desired.

To show this final property, differentiate Γ' to obtain

$$\Gamma''(x) = \widetilde{f}(\lfloor x \rfloor) - f(\lfloor x \rfloor),$$

where

$$f(x) = \binom{N}{x} q^x (1-q)^{N-x}$$

and

$$\widetilde{f}(x) = \binom{N+1}{x} (q')^x (1-q')^{N+1-x}.$$

The sign of $\Gamma''(x)$ for integer x is the same as the sign of

$$\phi(x) \equiv \log \frac{\widetilde{f}(x)}{f(x)} = \log \left\{ \frac{N+1}{N+1-x} \left(\frac{N}{N+1} \right)^x \left(\frac{1 - q\frac{N}{N+1}}{1-q} \right)^{N-x} \left(1 - q\frac{N}{N+1} \right) \right\}.$$

Differentiating this function twice yields

$$\phi''(x) = \frac{1}{(N+1-x)^2} > 0,$$

so ϕ is a strictly convex function. Further,

$$\phi(0) = \log \frac{\widetilde{f}(0)}{f(0)} = \log \frac{\widetilde{F}(0)}{F(0)} > 0.$$

Thus ϕ is either always positive, downcrosses zero once, or downcrosses and then upcrosses zero once. The integer truncation in the definition of Γ'' cannot increase the number of crossings, so Γ'' must also exhibit one of these three behaviors. Given that $\Gamma'(0) > 0$, if Γ'' does not cross zero at all, then $\Gamma'(x) > 0$ for all $x \in (0, N)$. If Γ'' crosses zero exactly once as a downcrossing, then either $\Gamma'(x) > 0$ for all x, or else Γ' crosses zero once. Finally, if Γ'' crosses zero twice, then Γ' crosses zero either zero, one, or two times. This establishes the desired upper bound on the number of crossings.

$\square$

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